Virtual Substitution

A more efficient way to eliminate quantifiers compared to FM, Section 6.2.1, in linear rational arithmetic was developed by R. Loos and V. Weispfenning (1993).

The method is also known as *test point method* or *virtual substitution method*. In contrast to FM, the method does not require CNF/DNF transformations of a prenex formula $\{\exists, \forall\} x_1 \ldots \{\exists, \forall\} x_n.\phi$.

Let $\phi[x, \vec{y}]$ be a quantifier-free formula of linear arithmetic in negation normal form containing the free variables $x, \vec{y}$ where all negation symbols are removed. Any quantifier free formula ϕ can be effectively and equivalently transformed in this form, see Section 6.2.1 and for the removal of the operator $\neg$ rule ElimNeg.

The linear inequations in ϕ can be transformed such that x is either isolated or does not occur in the inequation: $x \circ_i s_i(\vec{y})$ and $0 \circ_j s'_j(\vec{y})$ with $\circ_i, \circ_j \in \{\approx, \not\approx, <, \leq, >, \geq\}$, that is, ϕ us a formula built from linear inequations, $\wedge$ and $\vee$.

The goal of the virtual substitution method is to identify a finite set T of "test points", i.e., LA terms such that

$$\{\forall, \exists\}\vec{y}.\exists x.\phi[x, \vec{y}] \quad \text{iff} \quad \{\forall, \exists\}\vec{y}. \bigvee_{t \in T} \phi[x, \vec{y}]\,\{x \mapsto t\}.$$

Semantically, an existential quantifier represents an infinite disjunction over $\mathbb{Q}$. The goal of virtual substitution is to replace this infinite disjunction by a finite disjunction.

If the values of the variables $\vec{y}$ are determined by some arbitrary but fixed assignment β for the $\vec{y}$, then ϕ can be considered as a function $\phi_\beta : \mathbb{Q} \mapsto \{0, 1\}$ by

$$\phi_\beta(d) := \mathcal{A}_{\mathsf{LRA}}(\beta[x \mapsto d])(\phi)$$

for any $d \in \mathbb{Q}$. The value of each of the atoms $x \circ_i s_i[\vec{y}]$ changes only at $\mathcal{A}_{\mathsf{LRA}}(\beta)(s_i[\vec{y}])$, and the value of ϕ can only change if the value of one of its atoms changes. So ϕ_β is a piecewise constant function.

More precisely, the set of all $d \in \mathbb{Q}$ with $\phi_\beta(d) = 1$ is a finite union of intervals. The union may be empty, the individual intervals may be finite or infinite and open or closed.

Let

$$\text{dist}(\phi, x, \beta) = \begin{array}{l} \min\{\, |\mathcal{A}_{\text{LRA}}(\beta)(s_i[\vec{y}]) - \mathcal{A}_{\text{LRA}}(\beta)(s_j[\vec{y}])| \\ \text{where } \mathcal{A}_{\text{LRA}}(\beta)(s_i[\vec{y}]) \neq \mathcal{A}_{\text{LRA}}(\beta)(s_j[\vec{y}])\,\} \end{array}$$

the minimal distance between two differently interpreted terms of atoms $x \circ_i s_i[\vec{y}]$, $x \circ_j s_j[\vec{y}]$ in ϕ under β. Then each of the intervals has either length 0, i.e., it consists of one point, or its length is at least $\text{dist}(\phi, x, \beta)$.

The set of all values $d \in \mathbb{Q}$ of $\phi_\beta(d)$ can be considered either by traversing $\mathbb{Q}$ from $-\infty$ to $+\infty$ or the other way round. In the case of traversing from $-\infty$ to $+\infty$ if the set of all d for which $\phi_\beta(d) = 1$ is non-empty, then

(i) $\phi_\beta(d) = 1$ for all $d \circ \mathcal{A}_{\mathsf{LRA}}(\beta)(r[\vec{y}])$ for some $x \circ r[\vec{y}]$ occurring in ϕ, $\circ \in \{<, \leq\}$ or

(ii) there is some value $d \in \mathbb{Q}$ where the value of $\phi_\beta(d)$ switches from 0 to 1 when traversing from $-\infty$ to $+\infty$.

This observation can be used to construct a set of test points symbolically without considering β explicitly. It is sufficient to keep in mind that the values for the $\vec{y}$ are fixed and to use then the terms from ϕ as representatives for the values from $\mathbb{Q}$.

The start is a “sufficiently small” test point $r[\vec{y}]$ to take care of case (i). For case (ii), $\phi[x, \vec{y}]$ can only switch from 0 to 1 if one of the atoms switches from 0 to 1. Recall that after the initial transformations on ϕ, only positive boolean combinations of atoms and $\wedge$ and $\vee$ are left, which are monotonic with respect to truth values.

Atoms of the form $x \leq s_i[\vec{y}]$ and $x < s_i[\vec{y}]$ do not switch from 0 to 1 when x grows.

Atoms of the form $x \geq s_i[\vec{y}]$ and $x \approx s_i[\vec{y}]$ switch from 0 to 1 at $s_i[\vec{y}]$ hence $s_i[\vec{y}]$ is a test point.

Atoms of the form $x > s_i[\vec{y}]$ and $x \not\approx s_i[\vec{y}]$ switch from 0 to 1 “right after” $s_i[\vec{y}]$, hence $s_i[\vec{y}] + \varepsilon$ for some $0 < \varepsilon < \delta(\vec{y})$ is a test point.

If $r[\vec{y}]$ is sufficiently small and $0 < \varepsilon < \delta(\vec{y})$, then

$$\begin{aligned} T := \{r[\vec{y}]\} &\cup \{\, s_i[\vec{y}] \quad\;\; \mid \circ_i \in \{\geq, =\}\,\} \\ &\cup \{\, s_i[\vec{y}] + \varepsilon \mid \circ_i \in \{>, \neq\}\,\}. \end{aligned}$$

is a set of test points for atoms $x \circ_i s_i[\vec{y}]$.

However, it is not known how small $r[\vec{y}]$ has to be for case (i), and $\delta(\vec{y})$ for case (ii) is not known as well, because it is not effectively possible to consider all, infinitely many β explicitly.

The idea out the problem is to extend the LA language by further symbols ∞, and ε with the obvious intended meanings. Now it is straightforward to define T independently of β.

$$\begin{aligned} T := \{-\infty\} &\cup \{\, s_i[\vec{y}] \quad\quad\; \mid \circ_i \in \{\geq, =\}\,\} \\ &\cup \{\, s_i[\vec{y}] + \varepsilon \mid \circ_i \in \{>, \neq\}\,\}. \end{aligned}$$

But the semantics of LA is not defined with respect to the infinitesimals ∞, ε and all considerations leading to the above set T do not hold anymore, if ϕ contains occurrences of ∞ or ε.

Fortunately, the infinitesimals ∞ and ε vanish when substituted for some variable x.

$$(x < s(\vec{y}))\,\{x \mapsto -\infty\} := \lim_{r \to -\infty} (r < s(\vec{y})) = \top$$
$$(x \leq s(\vec{y}))\,\{x \mapsto -\infty\} := \lim_{r \to -\infty} (r \leq s(\vec{y})) = \top$$
$$(x > s(\vec{y}))\,\{x \mapsto -\infty\} := \lim_{r \to -\infty} (r > s(\vec{y})) = \bot$$
$$(x \geq s(\vec{y}))\,\{x \mapsto -\infty\} := \lim_{r \to -\infty} (r \geq s(\vec{y})) = \bot$$
$$(x \approx s(\vec{y}))\,\{x \mapsto -\infty\} := \lim_{r \to -\infty} (r \approx s(\vec{y})) = \bot$$
$$(x \not\approx s(\vec{y}))\,\{x \mapsto -\infty\} := \lim_{r \to -\infty} (r \not\approx s(\vec{y})) = \top$$

$$(x < s(\vec{y}))\,\{x \mapsto u + \varepsilon\} := \lim_{\varepsilon \to 0}(u + \varepsilon < s(\vec{y})) = (u < s(\vec{y}))$$
$$(x \leq s(\vec{y}))\,\{x \mapsto u + \varepsilon\} := \lim_{\varepsilon \to 0}(u + \varepsilon \leq s(\vec{y})) = (u < s(\vec{y}))$$
$$(x > s(\vec{y}))\,\{x \mapsto u + \varepsilon\} := \lim_{\varepsilon \to 0}(u + \varepsilon > s(\vec{y})) = (u \geq s(\vec{y}))$$
$$(x \geq s(\vec{y}))\,\{x \mapsto u + \varepsilon\} := \lim_{\varepsilon \to 0}(u + \varepsilon \geq s(\vec{y})) = (u \geq s(\vec{y}))$$
$$(x \approx s(\vec{y}))\,\{x \mapsto u + \varepsilon\} := \lim_{\varepsilon \to 0}(u + \varepsilon \approx s(\vec{y})) = \bot$$
$$(x \not\approx s(\vec{y}))\,\{x \mapsto u + \varepsilon\} := \lim_{\varepsilon \to 0}(u + \varepsilon \not\approx s(\vec{y})) = \top$$

The above test point set is constructed by considering a traversal of possible values for x from $-\infty$ to $+\infty$. Alternatively, x can be traversed from $+\infty$ to $-\infty$. In this case, the test points are

$$\begin{aligned} T' := \{+\infty\} &\cup \{\, s_i[\vec{y}] \quad\quad \mid \circ_i \in \{\leq, =\}\,\} \\ &\cup \{\, s_i[\vec{y}] - \varepsilon \mid \circ_i \in \{<, \neq\}\,\}. \end{aligned}$$

Infinitesimals are eliminated in a similar way as before.

In practice, both sets T and T' and eventually the smaller formula after substitution and simplification is considered. Similar to the FM decision procedure for formulas, a universally quantified formula $\forall x.\phi$, is replaced by $\neg\exists x.\neg\phi$. Then the inner negation is pushed downwards, and then the test point procedure is applied as in the case of an existential quantifier.

Note that in contrast to the FM procedure, no CNF/DNF transformation is required. Loos-Weispfenning quantifier elimination works on arbitrary positive formulas. So the CNF/DNF conversion blow up caused in FM quantifier elimination does not happen for virtual substitution. Therefore, the worst-case complexity of Loos-Weispfenning quantifier elimination significantly improves upon the worst-case complexity of FM.

However, the cost of computing a negation normal form remain.

Virtual Substitution Complexity

The number of test points is at most half of the number of atoms for some formula ϕ with $|\phi| = n$, so the formula resulting from the elimination of one variable, independent from the type of the quantifier, is at most quadratic, therefore $O(n^2)$ runtime.
A sequence of m quantifiers of the same kind, results in a multiplication of the formula size with n in each step, therefore $O(n^{m+1})$ runtime. This is the result of distributing existential quantifiers over disjunctions.

$$\begin{array}{ll} & \exists x_2\, \exists x_1.\ \phi[x_1, x_2, \vec{y}] \\ \leftrightarrow & \exists x_2.\left(\bigvee_{t_1 \in T_1} \phi[x_1, x_2, \vec{y}]\,\{x_1 \mapsto t_1\}\right) \\ \leftrightarrow & \bigvee_{t_1 \in T_1} (\exists x_2.\ \phi[x_1, x_2, \vec{y}]\,\{x_1 \mapsto t_1\}) \\ \leftrightarrow & \bigvee_{t_1 \in T_1} \bigvee_{t_2 \in T_2} (\phi[x_1, x_2, \vec{y}]\,\{x_1 \mapsto t_1\}\,\{x_2 \mapsto t_2\}) \end{array}$$

A sequence of m quantifier alternations $\exists\forall\exists\forall\ldots\exists$ turns the top-level disjunction after moving the inner negation into a top-level conjunction. An existential quantifier does not distribute over a conjunction, so the procedure needs $O(n^2)$ runtime for each step, therefore doubly exponential runtime in sum, $O(n^{2^m})$.

Simplex

The Simplex algorithm is the prime algorithm for solving optimization problems of systems of linear inequations over the rationals. For automated reasoning optimization at the level of conjunctions of inequations is not in focus. Rather, solvability of a set of linear inequations as a subproblem of some theory combination is the typical application. In this context the simplex algorithm is useful as well, due to its incremental nature. If an inequation $t \circ c$, $\circ \in \{\leq, \geq, <, >\}$, $t = \sum a_i x_i$, $a_i, c \in \mathbb{Q}$, is added to a set N of inequations where the simplex algorithm has already found a solution for N, the algorithm needs not to start from scratch. Instead it continues with the solution found for N. In practice, it turns out that then typically only few steps are needed to derive a solution for $N \cup \{t \circ d\}$ if it exists.

Firstly, the problem is rescritcted to non-strict inequations. Starting point is a set N (conjunction) of (non-strict) inequations of the form $(\sum_{x_j \in X} a_{i,j} x_j) \circ_i c_i$ where $\circ_i \in \{\geq, \leq\}$ for all i. Note that an equation $\sum a_i x_i = c$ can be encoded by two inequations $\{\sum a_i x_i \leq c, \sum a_i x_i \geq c\}$.

The variables occurring in N are assumed to be totally ordered by some ordering $\prec$. The ordering $\prec$ will eventualy guarantee termination of the simplex algorithm, see Definition 6.2.10 and Theorem 6.2.11 below. I assume the x_j to be all different, without loss of generality $x_j \prec x_{j+1}$, and I assume that all coefficients are normalized by the gcd of the $a_{i,j}$ for all j: if the gcd is different from 1 for one inequation, it is used for division of all coefficients of the inequation.

The goal is to decide whether there exists an assignment β from the x_j into $\mathbb{Q}$ such that

$$\mathrm{LRA}(\beta) \models \bigwedge_i [(\sum_{x_j \in X} a_{i,j} x_j) \circ_i c_i]$$

or equivalently, $\mathrm{LRA}(\beta) \models N$. So the x_j are free variables, i.e., placeholders for concrete values, i.e., existentially quantified.

The first step is to transform the set N of inequations into two disjoint sets E, B of equations and simple bounds, respectively. The set E contains equations of the form $y_i \approx \sum_{x_j \in X} a_{i,j} x_j$, where the y_i are fresh and the set B contains the respective simple bounds $y_i \circ_i c_i$. In case the original inequation from N was already a simple bound, i.e., of the form $x_j \circ_j c_j$ it is simply moved to B. If in N left hand sides of ineqations $(\sum_{x_j \in X} a_{i,j} x_j) \circ_i c_i$ are shared, it is sufficient to introduce one equation for the respective left hand side. The y_i are also part of the total ordering $\prec$ on all variables.

The two representations are equivalent:

$$\mathrm{LRA}(\beta) \models N$$

iff

$$\mathrm{LRA}(\beta[y_i \mapsto \beta(\textstyle\sum_{x_j \in X} a_{i,j} x_j)]) \models E$$

and

$$\mathrm{LRA}(\beta[y_i \mapsto \beta(\textstyle\sum_{x_j \in X} a_{i,j} x_j)]) \models B.$$

Given E and B a variable z is called *dependent* if it occurs on the left hand side of an equation in E, i.e., there is an equation $(z \approx \sum_{x_j \in X} a_{i,j} x_j) \in E$, and in case such a defining equation for z does not exist in E the variable z is called *independent*. Note that by construction the initial y_i are all dependent and do not occur on the right hand side of an equation.

Given a dependant variable x, an independent variable y, and a set of equations E, the *pivot* operation exchanges the roles of x, y in E where y occurs with non-zero coefficient in the defining equation of x. Let $(x \approx ay + t) \in E$ be the defining equation of x in E. When writing $(x \approx ay + t)$ for some equation, I always assume that $y \notin \mathrm{vars}(t)$. Let E' be E without the defining equation of x. Then

$$\mathrm{piv}(E, x, y) := \{y \approx \frac{1}{a}x + \frac{1}{-a}t\} \cup E'\{y \mapsto (\frac{1}{a}x + \frac{1}{-a}t)\}$$

Given an assignment β, an independent variable y, a rational value c, and a set of equations E then the *update* of β with respect to y, c, and E is

$$\mathrm{upd}(\beta, y, c, E) := \beta[y \mapsto c, \{x \mapsto \beta[y \mapsto c](t) \mid x \approx t \in E\}]$$

A Simplex problem state is a quintuple $(E; B; \beta; S; s)$ where E is a set of equations; B a set of simple bounds; β an assignment to all variables in E, B; S a set of derived bounds, and s the status of the problem with $s \in \{\top, \text{IV}, \text{DV}, \bot\}$. The state $s = \top$ indicates that $\text{LRA}(\beta) \models S$; the state $s = \text{IV}$ that potentially $\text{LRA}(\beta) \not\models x \circ c$ for some independent variable x, $x \circ c \in S$; the state $s = \text{DV}$ that $\text{LRA}(\beta) \models x \circ c$ for all independent variables x, $x \circ c \in S$, but potentially $\text{LRA}(\beta) \not\models x' \circ c'$ for some dependent variable x', $x' \circ c' \in S$; and the state $s = \bot$ that the problem is unsatisfiable.

The following states can be distinguished:

$(E; B; \beta_0; \emptyset; \top)$	is the start state for N and its transformation into E, B, and assignment $\beta_0(x) := 0$ for all $x \in \text{vars}(E \cup B)$
$(E; \emptyset; \beta; S; \top)$	is a final state, where $\text{LRA}(\beta) \models E \cup S$ and hence the problem is solvable
$(E; B; \beta; S; \bot)$	is a final state, where $E \cup B \cup S$ has no model

The important invariants of the simplex rules are:

(i) for every dependent variable there is exactly one equation in E defining the variable and

(ii) dependent variables do not occur on the right hand side of an equation,

(iii) $\mathrm{LRA}(\beta) \models E$

These invariants are maintained by a pivot (piv) or an update (upd) operation.

EstablishBound $(E; B \uplus \{x \circ c\}; \beta; S; \top) \Rightarrow_{\text{SIMP}}$
$(E; B; \beta; S \cup \{x \circ c\}; \text{IV})$

AckBounds $(E; B; \beta; S; s) \Rightarrow_{\text{SIMP}} (E; B; \beta; S; \top)$
if $\text{LRA}(\beta) \models S$, $s \in \{\text{IV}, \text{DV}\}$

FixIndepVar $(E; B; \beta; S; \text{IV}) \Rightarrow_{\text{SIMP}}$
$(E; B; \text{upd}(\beta, x, c, E); S; \text{IV})$
if $(x \circ c) \in S$, $\text{LRA}(\beta) \not\models x \circ c$, x independent

AckIndepBound $(E; B; \beta; S; \text{IV}) \Rightarrow_{\text{SIMP}} (E; B; \beta; S; \text{DV})$

if $\text{LRA}(\beta) \models x \circ c$, for all independent variables x with bounds $x \circ c$ in S

FixDepVar$\leq$ $(E; B; \beta; S; \text{DV}) \Rightarrow_{\text{SIMP}} (E'; B; \text{upd}(\beta, x, c, E'); S; \text{DV})$

if $(x \leq c) \in S$, x dependent, $\text{LRA}(\beta) \not\models x \leq c$, there is an independent variable y and equation $(x \approx ay + t) \in E$ where $(a < 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S)$ or $(a > 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S)$ and $E' := \text{piv}(E, x, y)$

FixDepVar$\geq$ $(E; B; \beta; S; \text{DV}) \Rightarrow_{\text{SIMP}} (E'; B; \text{upd}(\beta, x, c, E'); S; \text{DV})$

if $(x \geq c) \in S$, x dependent, $\text{LRA}(\beta) \not\models x \geq c$, there is an independent variable y and equation $(x \approx ay + t) \in E$ where $(a > 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S)$ or $(a < 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S)$ and $E' := \text{piv}(E, x, y)$

FailBounds $(E; B; \beta; S; \top) \Rightarrow_{\text{SIMP}} (E; B; \beta; S; \bot)$

if there are two contradicting bounds $x \leq c_1$ and $x \geq c_2$ in $B \cup S$ for some variable x

FailDepVar$\leq$ $(E; B; \beta; S; \text{DV}) \Rightarrow_{\text{SIMP}} (E; B; \beta; S; \bot)$

if $(x \leq c) \in S$, x dependent, $\text{LRA}(\beta) \not\models x \leq c$ and there is no independent variable y and equation $(x \approx ay + t) \in E$ where ($a < 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S$) or ($a > 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S$)

FailDepVar$\geq$ $(E; B; \beta; S; \text{DV}) \Rightarrow_{\text{SIMP}} (E; B; \beta; S; \bot)$

if $(x \geq c) \in S$, x dependent, $\text{LRA}(\beta) \not\models x \geq c$ and there is no independent variable y and equation $(x \approx ay + t) \in E$ where (if $a > 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S$) or (if $a < 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S$)

FailBounds

$$(E; B; \beta; S; \top) \;\Rightarrow_{\mathrm{SIMP}}\; (E; B; \beta; S; \bot)$$

if there are two contradicting bounds $x \leq c_1$ and $x \geq c_2$ in $B \cup S$ for some variable x

Example:
if $\{x \geq 5, x \leq 0\} \subseteq B \cup S$, then

$$(E; B; \beta; S; \top) \Rightarrow_{\mathrm{SIMP}} (E; B; \beta; S; \bot)$$

FailBounds

$$(E; B; \beta; S; \top) \;\Rightarrow_{\text{SIMP}}\; (E; B; \beta; S; \bot)$$

if there are two contradicting bounds $x \leq c_1$ and $x \geq c_2$ in $B \cup S$ for some variable x

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EstablishBound

$$(E; B \uplus \{x \circ c\}; \beta; S; \top) \Rightarrow_{\text{SIMP}} (E; B; \beta; S \cup \{x \circ c\}; \text{IV})$$

Example:

$$E := \left\{ \begin{array}{l} u \approx x + 2y, \\ v \approx x - y \end{array} \right\}, \quad \begin{array}{ll} B & := \{x \geq 0, y \leq -1, u \geq 1, v \geq 2, v \leq 3\} \\ \beta & := \{x \mapsto 0, y \mapsto 0, u \mapsto 0, v \mapsto 0\} \\ S & := \{\} \end{array}$$

$$(E; B; \beta; \{\}; \top) \Rightarrow_{\text{SIMP}} (E; B \setminus \{x \geq 0\}; \beta; \{x \geq 0\}; \text{IV})$$

EstablishBound

$$(E; B \uplus \{x \circ c\}; \beta; S; \top) \Rightarrow_{\text{SIMP}} (E; B; \beta; S \cup \{x \circ c\}; \text{IV})$$

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$$(E; B \uplus \{x \circ c\}; \beta; S; \top) \Rightarrow_{\text{SIMP}} (E; B; \beta; S \cup \{x \circ c\}; \text{IV})$$

Example:

$$E := \left\{ \begin{array}{l} u \approx x + 2y, \\ v \approx x - y \end{array} \right\}, \begin{array}{ll} B & := \{x \geq 0, y \leq -1, u \geq 1, v \geq 2, v \leq 3\} \\ \beta & := \{x \mapsto 0, y \mapsto 0, u \mapsto 0, v \mapsto 0\} \\ S & := \{\} \end{array}$$

$$(E; B; \beta; \{\}; \top) \quad \Rightarrow_{\text{SIMP}} \quad (E; B \setminus \{x \geq 0\}; \beta; \{x \geq 0\}; \text{IV})$$

AckBounds

$$(E; B; \beta; S; V) \Rightarrow_{\text{SIMP}} (E; B; \beta; S; \top)$$

if $\text{LRA}(\beta) \models S$, $V \in \{\text{IV}, \text{DV}\}$

Example:

$$E := \left\{ \begin{array}{l} u \approx x + 2y, \\ v \approx x - y \end{array} \right\}, \quad \begin{array}{ll} B & := \{y \leq -1, u \geq 1, v \geq 2, v \leq 3\} \\ \beta & := \{x \mapsto 0, y \mapsto 0, u \mapsto 0, v \mapsto 0\} \\ S & := \{x \geq 0\} \end{array}$$

$$\begin{array}{lll} (E; B; \beta; S; \text{IV}) & \Rightarrow_{\text{SIMP}} & (E; B; \beta; S; \top) \\ & \Rightarrow_{\text{SIMP}} & (E; B \setminus \{y \leq -1\}; \beta; S \cup \{y \leq -1\}; \text{IV}) \end{array}$$

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$$(E; B; \beta; S; V) \Rightarrow_{\text{SIMP}} (E; B; \beta; S; \top)$$

if $\text{LRA}(\beta) \models S$, $V \in \{\text{IV}, \text{DV}\}$

Example:

$$E := \left\{ \begin{array}{l} u \approx x + 2y, \\ v \approx x - y \end{array} \right\}, \quad \begin{array}{ll} B & := \{y \leq -1, u \geq 1, v \geq 2, v \leq 3\} \\ \beta & := \{x \mapsto 0, y \mapsto 0, u \mapsto 0, v \mapsto 0\} \\ S & := \{x \geq 0\} \end{array}$$

$$\begin{array}{lll} (E; B; \beta; S; \text{IV}) & \Rightarrow_{\text{SIMP}} & (E; B; \beta; S; \top) \\ & \Rightarrow_{\text{SIMP}} & (E; B \setminus \{y \leq -1\}; \beta; S \cup \{y \leq -1\}; \text{IV}) \end{array}$$

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FixIndepVar

$$(E; B; \beta; S; \mathrm{IV}) \Rightarrow_{\mathrm{SIMP}} (E; B; \mathrm{upd}(\beta, x, c, E); S; \mathrm{IV})$$

if $(x \circ c) \in S$, $\mathrm{LRA}(\beta) \not\models x \circ c$, x independent

Example:

$$E := \left\{ \begin{array}{l} u \approx x + 2y, \\ v \approx x - y \end{array} \right\}, \quad \begin{array}{ll} B & := \{u \geq 1, v \geq 2, v \leq 3\} \\ \beta & := \{x \mapsto 0, y \mapsto 0, u \mapsto 0, v \mapsto 0\} \\ S & := \{x \geq 0, y \leq -1\} \end{array}$$

$$\begin{array}{ll} \beta' & := \mathrm{upd}(\beta, y, -1, E) \\ & := \{x \mapsto 0, y \mapsto -1, u \mapsto (0 + 2 * (-1)), v \mapsto (0 - (-1))\} \\ & := \{x \mapsto 0, y \mapsto -1, u \mapsto -2, v \mapsto 1\} \end{array}$$

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FixDepVar$\geq$

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if $(x \geq c) \in S$, x dependent, $\mathrm{LRA}(\beta) \not\models x \geq c$, there is an independent variable y and equation $(x \approx ay + t) \in E$ where $(a > 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S)$ or $(a < 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S)$ and $E' := \mathrm{piv}(E, x, y)$

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if $(x \leq c) \in S$, x dependent, $\text{LRA}(\beta) \not\models x \leq c$ and there is no independent variable y and equation $(x \approx ay + t) \in E$ where $(a < 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S)$ or $(a > 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S)$

Example:

$$E := \left\{ \begin{array}{l} x \approx u - 2y, \\ v \approx u - 3y \end{array} \right\}, \begin{array}{ll} B & := \{v \geq 2\} \\ \beta & := \{u \mapsto 1, y \mapsto -1, x \mapsto 3, v \mapsto 4\} \\ S & := \{x \geq 0, y \leq -1, u \geq 1, v \leq 3\} \end{array}$$

FailDepVar$\geq$

$$(E; B; \beta; S; \mathrm{DV}) \ \Rightarrow_{\mathrm{SIMP}} \ (E; B; \beta; S; \bot)$$

if $(x \geq c) \in S$, x dependent, $\beta \not\models_{\mathrm{LA}} x \geq c$ and there is no independent variable y and equation $(x \approx ay + t) \in E$ where (if $a > 0$ and $\beta(y) < c'$ for all $(y \leq c') \in S$) or (if $a < 0$ and $\beta(y) > c'$ for all $(y \geq c') \in S$)

6.2.7 Lemma (Simplex State Invariants)

The following invariants hold for any state $(E_i; B_i; \beta_i; S_i; s_i)$ derived by $\Rightarrow_{\text{SIMP}}$ on a start state $(E_0; B_0; \beta_0; \emptyset; \top)$:

(i) for every dependent variable there is exactly one equation in E defining the variable

(ii) dependent variables do not occur on the right hand side of an equation

(iii) $\text{LRA}(\beta) \models E_i$

(iv) for all independant variables x either $\beta_i(x) = 0$ or $\beta_i(x) = c$ for some bound $x \circ c \in S_i$

(v) for all assignments α it holds $\text{LRA}(\alpha) \models E_0$ iff $\text{LRA}(\alpha) \models E_i$

6.2.8 Lemma (Simplex Run Invariants)

For any run of $\Rightarrow_{\text{SIMP}}$ from start state $(E_0; B_0; \beta_0; \emptyset; \top) \Rightarrow_{\text{SIMP}} (E_1; B_1; \beta_1; S_1; s_1) \Rightarrow_{\text{SIMP}} \ldots$:

(i) the set $\{\beta_o, \beta_1, \ldots\}$ is finite

(ii) if the sets of dependent and independent variables for two equational systems E_i, E_j coincide, then $E_i = E_j$

(iii) the set $\{E_o, E_1, \ldots\}$ is finite

(iv) let S_i not contain contradictory bounds, then $(E_i; B_i; \beta_i; S_i; s_i) \Rightarrow_{\text{SIMP}}^{\text{FIV},*}$ is finite

6.2.9 Corollary (Infinite Runs Contain a Cycle)

Let $(E_0; B_0; \beta_0; \emptyset; \top) \Rightarrow_{\text{SIMP}} (E_1; B_1; \beta_1; S_1; s_1) \Rightarrow_{\text{SIMP}} \ldots$ be an infinite run. Then there are two states $(E_i; B_i; \beta_i; S_i; s_i)$, $(E_k; B_k; \beta_k; S_k; s_k)$ such that $i \neq k$ and $(E_i; B_i; \beta_i; S_i; s_i) = (E_k; B_k; \beta_k; S_k; s_k)$.

6.2.10 Definition (Reasonable Strategy)

A *reasonable* strategy prefers FailBounds over EstablishBounds and the FixDepVar rules select minimal variables x, y in the ordering $\prec$.

6.2.11 Theorem (Simplex Soundness, Completeness & Termination)

Given a reasonable strategy and initial set N of inequations and its separation into E and B :

(i) $\Rightarrow_{\text{SIMP}}$ terminates on $(E; B; \beta_0; \emptyset; \top)$,

(ii) if $(E; B; \beta_0; \emptyset; \top) \Rightarrow^*_{\text{SIMP}} (E'; B'; \beta; S; \bot)$ then N has no solution,

(iii) if $(E; B; \beta_0; \emptyset; \top) \Rightarrow^*_{\text{SIMP}} (E'; \emptyset; \beta; B; \top)$ and $(E; \emptyset; \beta; B; \top)$ is a normal form, then $\text{LRA}(\beta) \models N$,

(iv) all final states $(E'; B'; \beta; S; s)$ match either (ii) or (iii).