Orient $(E \uplus \{s \dot{\approx} t\}; R) \Rightarrow_{\text{KBC}} (E; R \cup \{s \to t\})$
if $s \succ t$

Delete $(E \uplus \{s \approx s\}; R) \Rightarrow_{\text{KBC}} (E; R)$

Deduce $(E; R) \Rightarrow_{\text{KBC}} (E \cup \{s \approx t\}; R)$
if $\langle s, t \rangle \in \text{cp}(R)$

Simplify-Eq $(E \uplus \{s \dot{\approx} t\}; R) \ \Rightarrow_{\text{KBC}} \ (E \cup \{u \approx t\}; R)$
if $s \rightarrow_R u$

R-Simplify-Rule $(E; R \uplus \{s \rightarrow t\}) \ \Rightarrow_{\text{KBC}} \ (E; R \cup \{s \rightarrow u\})$
if $t \rightarrow_R u$

L-Simplify-Rule $(E; R \uplus \{s \rightarrow t\}) \ \Rightarrow_{\text{KBC}} \ (E \cup \{u \approx t\}; R)$
if $s \rightarrow_R u$ using a rule $l \rightarrow r \in R$ so that $s \sqsupset l$, see below.

Trivial equations cannot be oriented and since they are not needed they can be deleted by the Delete rule.

The rule Deduce turns critical pairs between rules in R into additional equations. Note that if $\langle s, t \rangle \in \mathrm{cp}(R)$ then $s_R \leftarrow u \rightarrow_R t$ and hence $R \models s \approx t$.

The simplification rules are not needed but serve as reduction rules, removing redundancy from the state. Simplification of the left-hand side may influence orientability and orientation of the result. Therefore, it yields an equation. For technical reasons, the left-hand side of $s \rightarrow t$ may only be simplified using a rule $l \rightarrow r$, if $l \rightarrow r$ cannot be simplified using $s \rightarrow t$, that is, if $s \sqsupset l$, where the *encompassment quasi-ordering* $\sqsupseteq$ is defined by $s \sqsupseteq l$ if $s|_p = l\sigma$ for some p and σ and $\sqsupset = \sqsupseteq \setminus \sqsubseteq$ is the strict part of $\sqsupseteq$.

4.4.4 Proposition (Knuth-Bendix Completion Correctness)

If the completion procedure on a set of equations E is run, different things can happen:

1. A state where no more inference rules are applicable is reached and E is not empty. $\Rightarrow$ Failure (try again with another ordering?)
2. A state where E is empty is reached and all critical pairs between the rules in the current R have been checked.
3. The procedure runs forever.

4.4.5 Definition (Run)

A (finite or infinite) sequence $(E_0; R_0) \Rightarrow_{KBC} (E_1; R_1) \Rightarrow_{KBC} (E_2; R_2) \Rightarrow_{KBC} \ldots$ with $R_0 = \emptyset$ is called a *run* of the completion procedure with input E_0 and $\succ$. For a run, $E_\infty = \bigcup_{i \geq 0} E_i$ and $R_\infty = \bigcup_{i \geq 0} R_i$.

4.4.6 Definition (Persistent Equations)

The sets of *persistent equations of rules* of the run are $E_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} E_j$ and $R_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} R_j$.

Note: If the run is finite and ends with E_n, R_n then $E_* = E_n$ and $R_* = R_n$.

4.4.7 Definition (Fair Run)

A run is called *fair* if $\mathrm{CP}(R_*) \subseteq E_\infty$ (i.e., if every critical pair between persisting rules is computed at some step of the derivation).

4.4.10 Theorem (KBC Soundness)

Let $(E_0; R_0) \Rightarrow_{KBC} (E_1; R_1) \Rightarrow_{KBC} (E_2; R_2) \Rightarrow_{KBC} \ldots$ be a fair run and let R_0 and E_* be empty. Then

1. every proof in $E_\infty \cup R_\infty$ is equivalent to a rewrite proof in R_*,
2. R_* is equivalent to E_0 and
3. R_* is convergent.

Complexity

3.15.2 Theorem (Equational Logic Validity is Undecidable)

Validity of an equation modulo a set of equations is undecidable.

(Proof Scetch) Given a PCP with word lists $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ over alphabet $\{a, b\}$, it is represented by two unary functions g_a and g_b, constants ϵ, c, d, and a binary function f_R , all over some sort S. Then a word pair u_i, v_i is encoded by the equation $f_R(u_i(x), v_i(y)) \approx f_R(x, y)$ and the start state with the empty word is encoded by equation $f_R(\epsilon, \epsilon) \approx d$ and the final state identifying two equal words different from ϵ by the equations $f_R(g_a(x), g_a(x)) \approx c, \quad f_R(g_b(x), g_b(x)) \approx c$. I call the set of these equations E. Now the PCP over the two word lists has a solution iff $E \models c \approx d$.

4.4.11 Corollary (KBC Termination)

Termination of $\Rightarrow_{KBC}$ is undecidable for some given finite set of equations E.

(Proof Scetch) Using exactly the construction of Theorem 3.15.2 it remains to be shown that all computed critical pairs can be oriented. Critical pairs corresponding to the search for a PCP solution result in equations $f_R(u(x), v(y)) \approx f_R(u'(x), v'(y))$ or $f_R(u'(x), v'(x)) \approx c$. By chosing an appropriate ordering, all these equations can be oriented. Thus $\Rightarrow_{KBC}$ does not produce any unorientable equations. The rest follows from Theorem 3.15.2.

Unfailing Completion

Classical completion: Try to transform a set E of equations into an equivalent convergent TRS. Fail, if an equation can neither be oriented nor deleted.

Unfailing completion: If an equation cannot be oriented, *orientable instances* can still be used for rewriting. Note: If $\succ$ is total on ground terms, then every *ground instance* of an equation is trivial or can be oriented. The goal is to derive a *ground convergent* set of equations.

Let E be a set of equations, let $\succ$ be a reduction ordering. The relation $\rightarrow_{E^\succ}$ is defined by

$$s \rightarrow_{E^\succ} t \quad \text{iff} \quad \begin{array}{l} \text{there exist } (u \approx v) \in E \text{ or } (v \approx u) \in E, \\ p \in pos(s), \text{ and } \sigma : \mathcal{X} \rightarrow T(\Sigma, \mathcal{X}), \\ \text{so that } s|_p = u\sigma \text{ and } t = s[v\sigma]_p \\ \text{and } u\sigma \succ v\sigma. \end{array}$$

Note: $\rightarrow_{E^\succ}$ is terminating by construction.
From now on let $\succ$ be a reduction ordering that is total on ground terms.

E is called ground convergent w.r.t. $\succ$, if for all ground terms s and t with $s \leftrightarrow_E^* t$ there exists a ground term v so that $s \rightarrow_{E^\succ}^* v \; {}_{E^\succ}^{*}\!\!\leftarrow t$. (Analogously for $E \cup R$.)
As for standard completion, ground convergence is established by computing critical pairs.
However, the ordering $\succ$ is not total on non-ground terms. Since $s\theta \succ t\theta$ implies $s \not\preceq t$, $\succ$ is approximated on ground terms by $\not\preceq$ on arbitrary terms.
Let $u_i \doteq v_i$ ($i = 1, 2$) be equations in E whose variables have been renamed so that $vars(u_1 \doteq v_1) \cap vars(u_2 \doteq v_2) = \emptyset$. Let $p \in pos(u_1)$ be a position so that $u_1|_p$ is not a variable, σ is an mgu of $u_1|_p$ and u_2, and $u_i\sigma \not\preceq v_i\sigma$ ($i = 1, 2$). Then $\langle v_1\sigma, (u_1\sigma)[v_2\sigma]_p \rangle$ is called a *semi-critical pair* of E with respect to $\succ$.

The set of all semi-critical pairs of E is denoted by $\mathrm{sp}_{\succ}(E)$.
Semi-critical pairs of $E \cup R$ are defined analogously. If $\rightarrow_R \subseteq \succ$, then $\mathrm{cp}(R)$ and $\mathrm{sp}_{\succ}(R)$ agree.
Note: In contrast to critical pairs, it may be necessary to consider overlaps of a rule with itself at the top. For instance, if $E = \{f(x) \approx g(y)\}$, then $\langle g(y), g(y')\rangle$ is a non-trivial semi-critical pair.
The *Deduce* rule takes now the following form:

Deduce $\quad (E; R) \;\Rightarrow_{\mathrm{UKBC}}\; (E \cup \{s \approx t\}; R)$
if $\langle s, t\rangle \in \mathrm{sp}_{\succ}(E \cup R)$
The other rules are inherited from $\Rightarrow_{KBC}$. Moreover, the fairness criterion for runs is replaced by

$$\mathrm{sp}_{\succ}(E_* \cup R_*) \subseteq E_\infty$$

(i.e., if every semi-critical pair between persisting rules or equations is computed at some step of the derivation).

Analogously to Theorem 4.4.10 now the following theorem is obtained:

4.4.12 Theorem (Convergence)

Let $(E_0, R_0) \Rightarrow_{\text{UKBC}} (E_1, R_1) \Rightarrow_{\text{UKBC}} (E_2, R_2) \Rightarrow \ldots$ be a fair run; let $R_0 = \emptyset$. Then

1. $E_* \cup R_*$ is equivalent to E_0, and
2. $E_* \cup R_*$ is ground convergent.

Moreover one can show that, whenever there exists a *reduced* convergent R so that $\approx_{E_0} = \downarrow_R$ and $\rightarrow_R \in \succ$, then for every fair *and simplifying* run $E_* = \emptyset$ and $R_* = R$ up to variable renaming.

Here R is called reduced, if for every $l \rightarrow r \in R$, both l and r are irreducible w.r.t. $R \setminus \{l \rightarrow r\}$. A run is called simplifying, if R_* is reduced, and for all equations $u \approx v \in E_*$, u and v are incomparable w.r.t. $\succ$ and irreducible w.r.t. R_*.

Unfailing completion is refutationally complete for equational theories: