Finding Fair and Efficient Allocations

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Computing Pareto-Optimal and Almost Envy-Free Allocations of Indivisible Goods

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presented by Tim Göttlicher

Finding **fair** and **efficient** allocations

indivisible goods

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?

indivisible goods

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EF1

indivisible goods

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Envy-free
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Maximize Nash welfare

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NP-hard

~~Maximize Nash welfare~~

polynomial time?

indivisible goods

Finding **fair** and **efficient** allocations

EF1

Envy-free
up to one item

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Pareto-optimal

Algorithm?

NP-hard

~~Maximize Nash welfare~~

pseudo-polynomial time?

EF1 – Envy-freeness up to one item

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$\forall a, b$ agents $\exists g$ good

$$V_a(x_a) \geq V_a(x_b \setminus g)$$

own bundle

other bundle without g

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own bundle

other bundle
without g

PO – Pareto-optimality

there is **no** feasible allocation y such that

$$\forall a \quad V_a(y_a) \geq V_a(x_a)$$

$$\exists a \quad V_a(y_a) > V_a(x_a)$$

y dominates x

Market prices as a common denominator

Agents have different preferences

how to satisfy everyone at the same time?

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Economics: coordination via markets

$\rightarrow$ prices are the same for everyone

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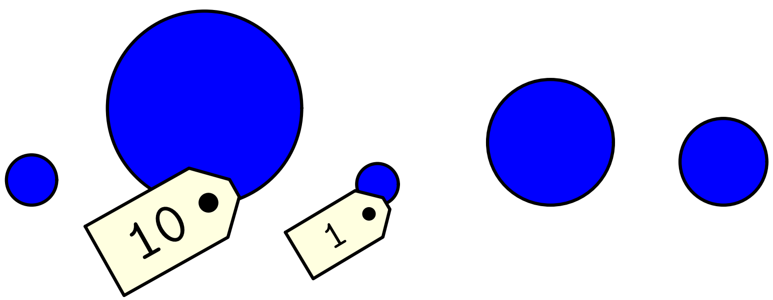

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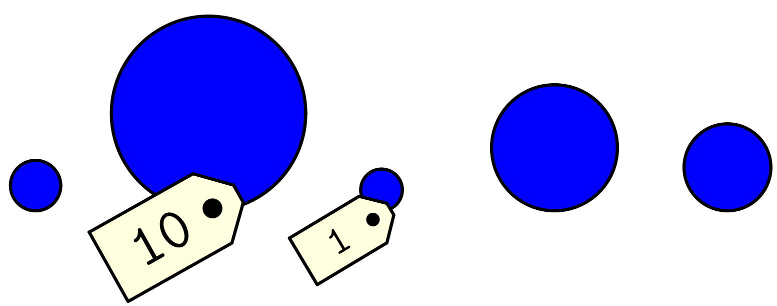

need relationship between price and value

Maximum bang-per-buck allocation

Bang-per-buck

$$\frac{\text{utility}}{\text{price}} \quad \frac{v_{ag}}{p_g}$$

Maximum bang-per-buck allocation

Bang-per-buck

$$\frac{\text{utility}}{\text{price}} \quad \frac{v_{ag}}{p_g} \longrightarrow$$

MBB – Maximum bang-per-buck

$$\mathrm{mbb}_a = \max_g \frac{v_{ag}}{p_g}$$

Maximum bang-per-buck allocation

Bang-per-buck

$$\frac{\text{utility}}{\text{price}} \quad \frac{v_{ag}}{p_g} \longrightarrow$$

MBB – Maximum bang-per-buck

$$\text{mbb}_a = \max_g \frac{v_{ag}}{p_g}$$

MBB allocation

all agents are allocated only items that are MBB for them

$$\forall a \; \forall g \in x_a, \quad \frac{v_{ag}}{p_g} = \text{mbb}_a$$

How to find a MBB allocation

utility	v	g_1	g_2	g_3	g_4	g_5
	a	1	2	1	2	1
	b	7	4	6	2	0
	c	8	4	2	4	5

How to find a MBB allocation

1. allocate item g to agent a with max $v_{a,g}$

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$\frac{v}{p}$	g_1	g_2	g_3	g_4	g_5
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≤ 1

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≤ 1

mbb$_c$

MBB bounds bundle utility to price

$$\mathrm{mbb}_a = \max_g \frac{v_{ag}}{p_g} \qquad\qquad v_{a,g} \geq 0 \qquad\qquad p_g > 0$$

Assumption: Linear utilities

$$V_a(y) = \sum_g v_{a,g} y_g$$

Lemma

$$V_a(y) \leq \mathrm{mbb}_a P(y) \qquad \text{any bundle } y$$

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$V_a(y) \leq \mathrm{mbb}_a P(y)$ any bundle y

$V_a(x_a) = \mathrm{mbb}_a P(x_a)$ if x is MBB with prices P

Complete MBB allocations are Pareto-Optimal

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there is **no** feasible allocation y that dominates x

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$$\forall a \quad V_a(y_a) \geq V_a(x_a)$$

$$\exists b \quad V_b(y_b) > V_b(x_b)$$

cannot exist: sum of prices would increase

$(\mathbf{x}, \mathbf{p})$ is a **Fisher market equilibrium** if it is

1. Market clearing

2. On MBB goods

3. Budget exhausting

allocation prices

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3. Budget exhausting

$$\forall a \sum_g x_{a,g} p_g = e_a$$

Equal budgets imply Envy-Freeness

Lemma

$$V_a(y) \leq \mathrm{mbb}_a P(y) \qquad \text{any bundle } y$$

$$V_a(x_a) = \mathrm{mbb}_a P(x_a) \qquad \text{if } x \text{ is MBB with prices } P$$

Assume - x is MBB allocation

- $\forall a, P(x_a) = e$ (equal budgets)

Equal budgets imply Envy-Freeness

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- $\forall a, P(x_a) = e$ (equal budgets)

$\implies$ $\forall a, b \quad V_a(x_a) \geq V_a(x_b)$ (envy-free)

EF1 transferred to spending

equal budgets $\implies$ EF

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not possible in indivisible goods

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EF1 – Envy-freeness up to one item

$\forall a, b$ agents $\exists g$ good

$$V_a(x_a) \geq V_a(x_b \setminus g)$$

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transfer to budgets

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EF1 – Envy-freeness up to one item

$\forall a, b$ agents $\exists g$ good

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transfer to budgets

pEF1 – Price envy-freeness up to one item

$\forall a, b$ agents $\exists g$ good

$$P(x_a) \geq P(x_b \setminus g)$$

pEF1 implies EF1 on MBB allocation

Lemma

$$V_a(y) \leq \mathrm{mbb}_a P(y)$$ any bundle y

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- $\forall a, b \; \exists g$
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Assume
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- $\forall a, b\ \exists g$
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$\implies$ $V_a(x_a) \geq V_a(x_b \setminus g)$ (EF1)

$(\mathbf{x}, \mathbf{p})$ is a **Fisher market equilibrium** if it is

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$$\forall g \sum_a x_{a,g} = 1$$

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$$\forall a \ \forall g \in x_a, \quad \frac{v_{ag}}{p_g} = \mathrm{mbb}_a = \max_g \frac{v_{ag}}{p_g}$$

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PO

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$\forall a, b$ agents $\quad \exists g$ good

$$P(x_a) \geq P(x_b \setminus g)$$

PO

EF1

Overview

$$\text{Complete MBB allocation} \implies \text{Pareto optimal}$$

$$\text{MBB} + \text{pEF1} \implies \text{EF1}$$

Overview

$$\text{Complete MBB allocation} \implies \text{Pareto optimal}$$

$$\text{MBB} + \text{pEF1} \implies \text{EF1}$$

Algorithm

initial MBB allocation

maintain MBB

swap items, change prices

pEF1

Algorithm ideas

pEF1

$\forall a, b$ agents $\quad \exists g$ good

$$P(x_a) \geq P(x_b \setminus g)$$

Algorithm ideas

pEF1

$\forall a, b$ agents $\quad \exists g$ good

$$P(x_a) \geq P(x_b \setminus g)$$

$$P_{\min} \geq P(x_b \setminus g)$$

Algorithm ideas

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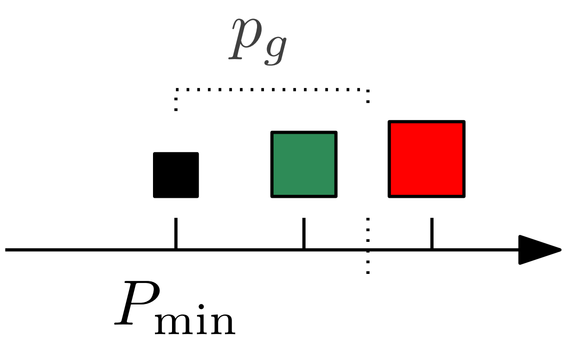

Algorithm ideas

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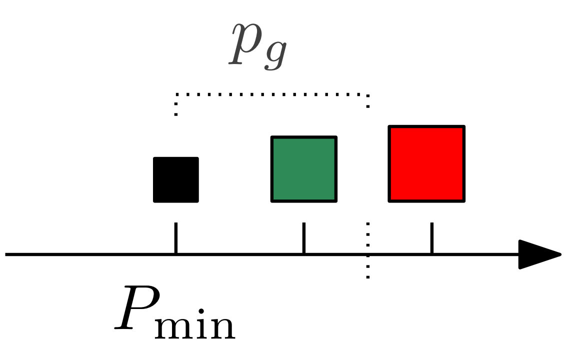

increase $P_{\min}$, reduce prices $\longrightarrow$ progress towards pEF1

Algorithm ideas

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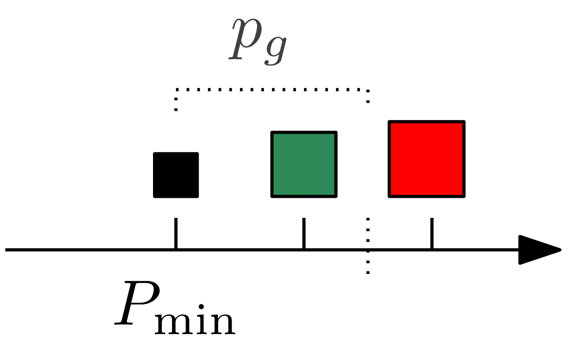

swap items to least spender

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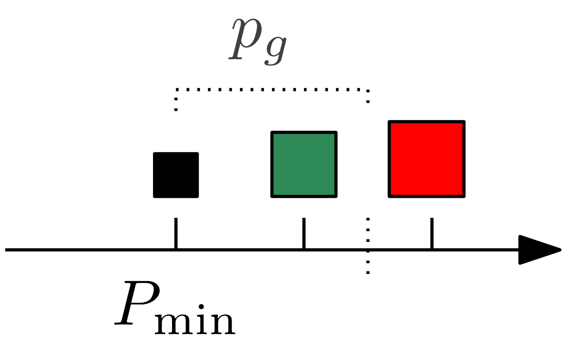
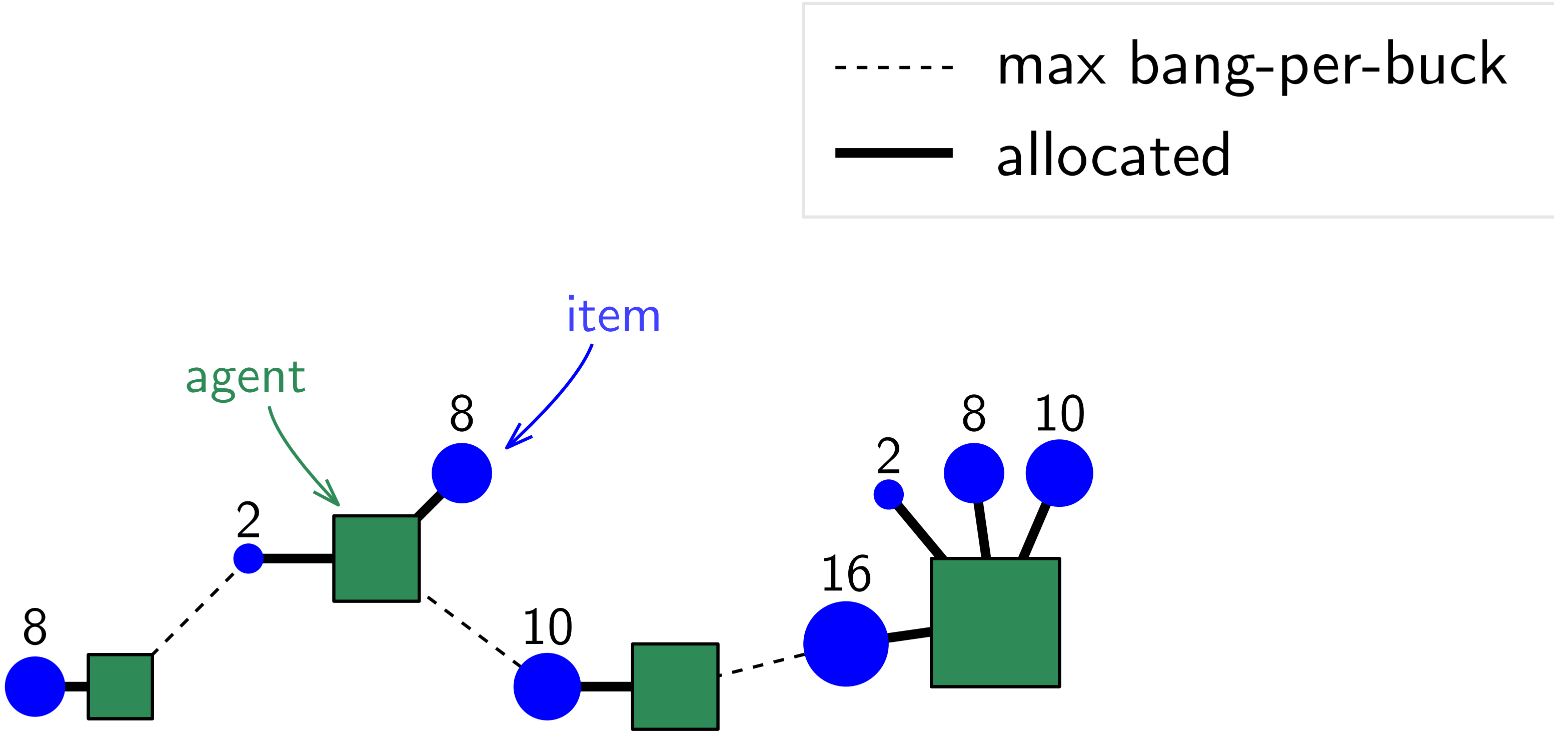


swap items to least spender

increase $P_{\min}$, reduce prices $\longrightarrow$ progress towards pEF1

maintain $\quad \forall a \;\; x_a \subseteq \mathrm{MBB}_a$

Allocation and bang-per-buck as a graph



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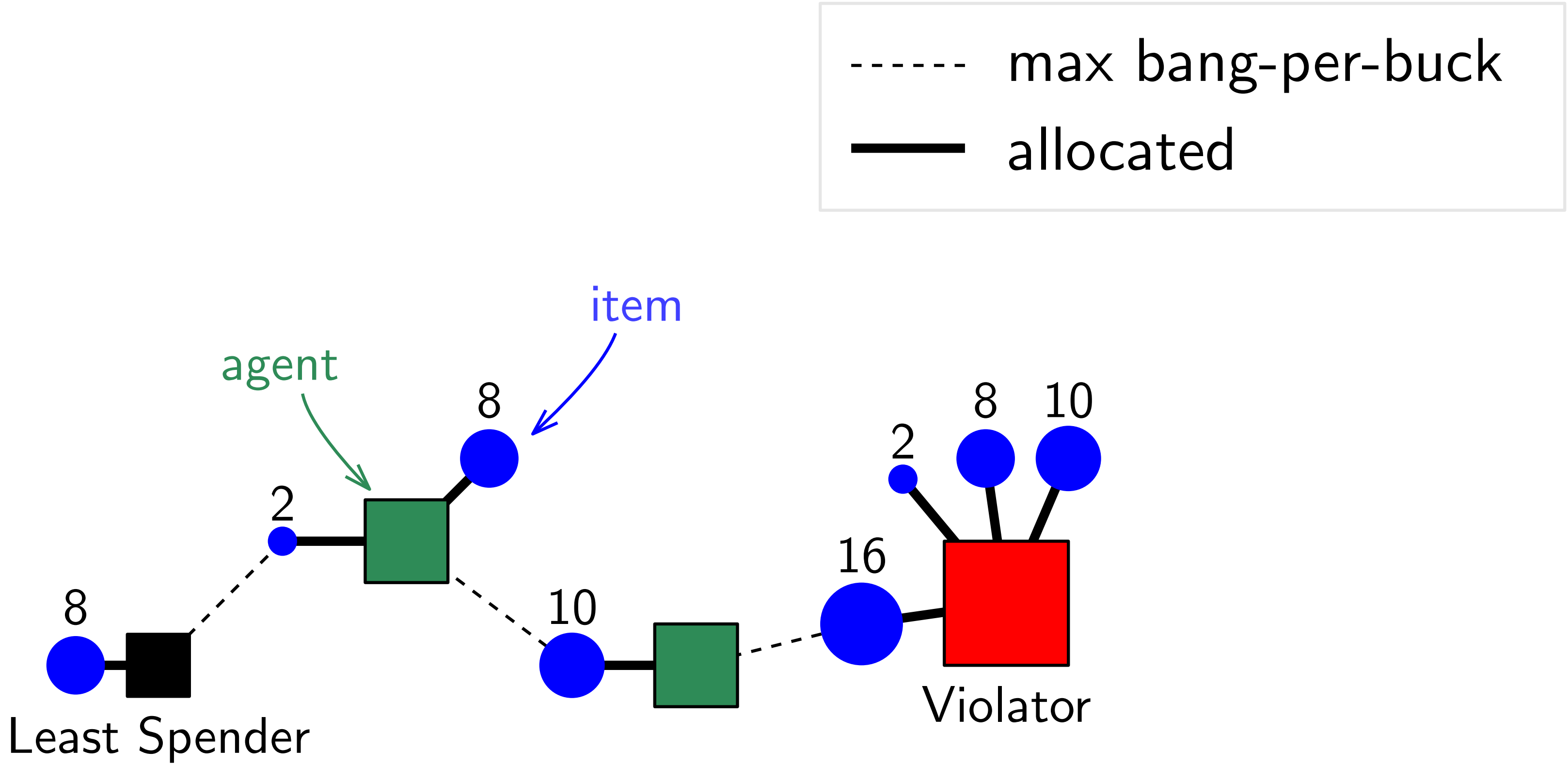

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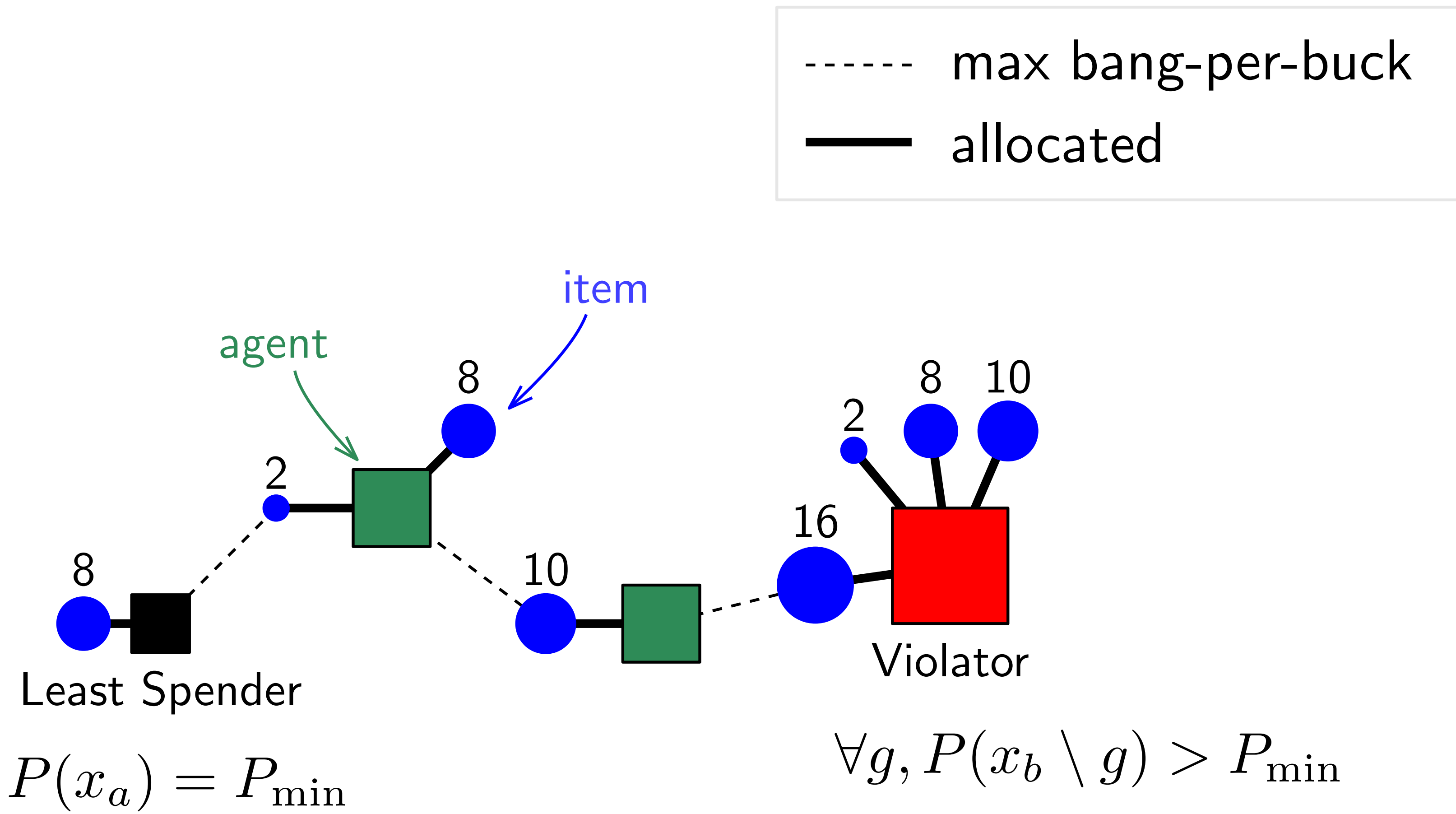
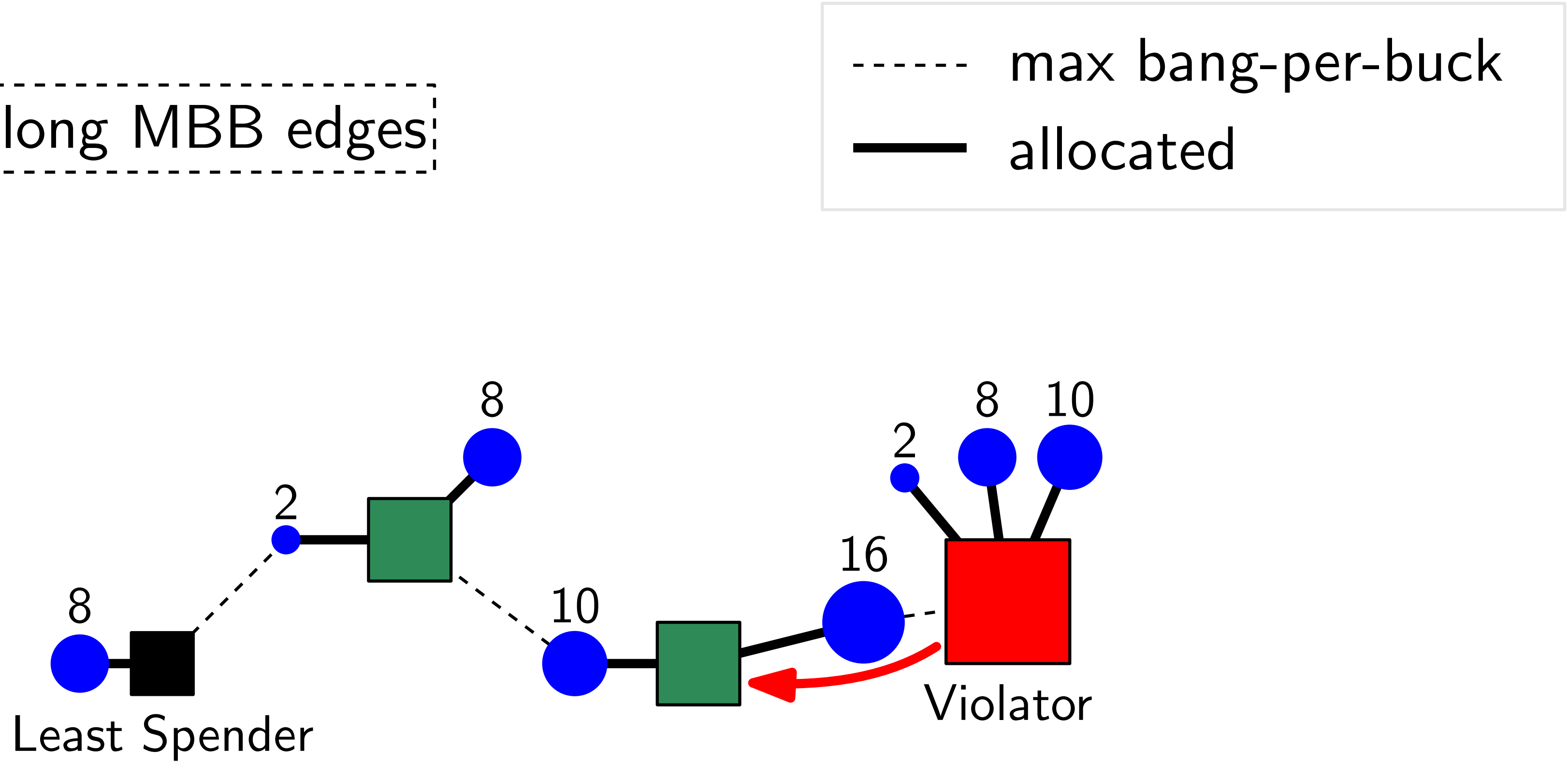


Allocation and bang-per-buck as a graph

Swaps along MBB edges



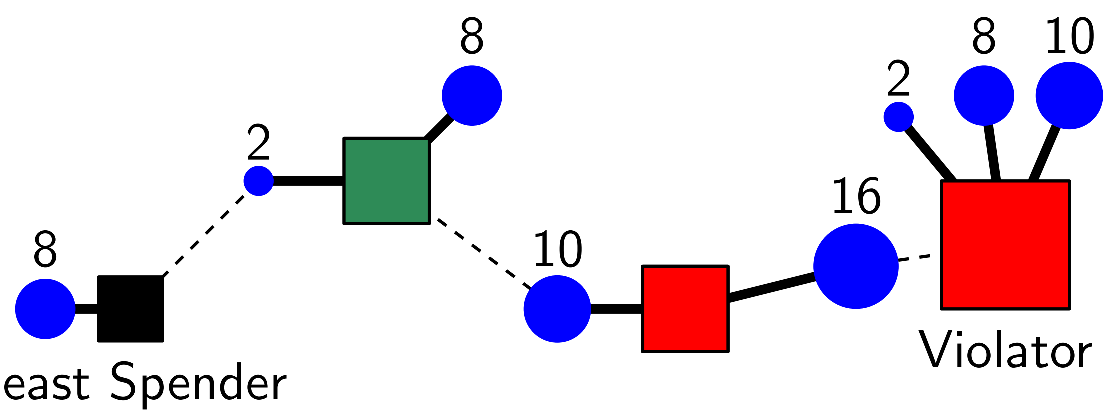
$P(x_a) = P_{\min}$

$\forall g, P(x_b \setminus g) > P_{\min}$

Allocation and bang-per-buck as a graph

Swaps along MBB edges

- - - - max bang-per-buck
▬▬ allocated



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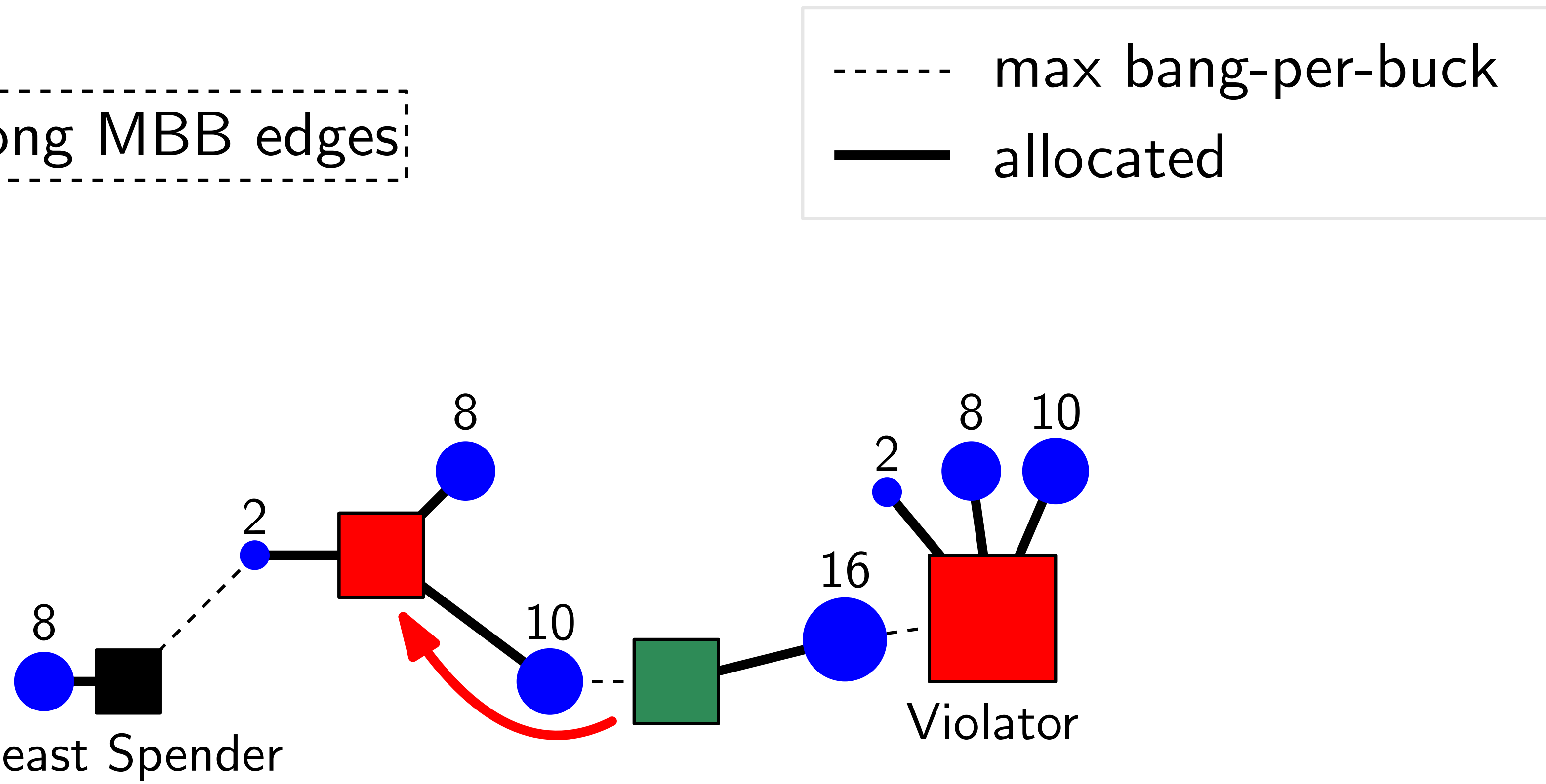

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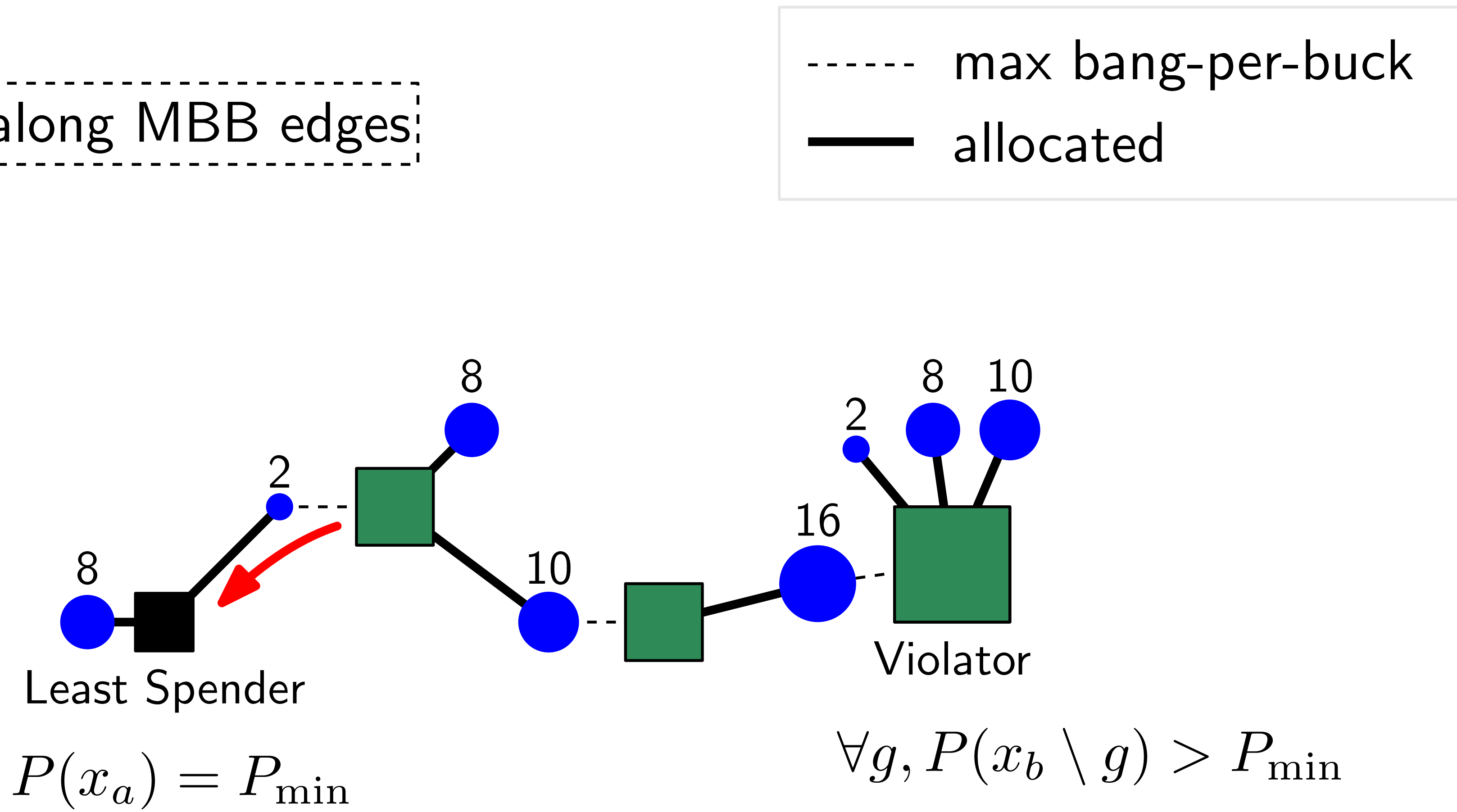


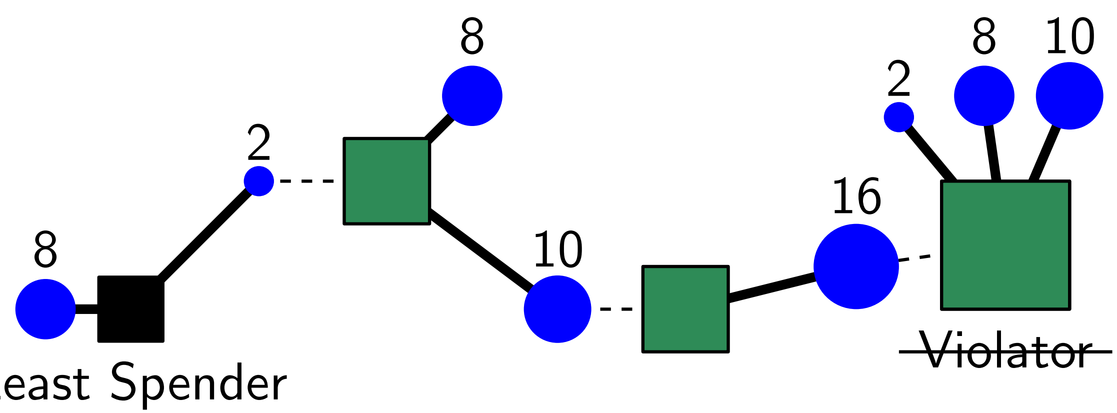
Allocation and bang-per-buck as a graph

Swaps $\boxed{\text{along MBB edges}}$

No violators left $\implies$ pEF1

- - - - - - max bang-per-buck

——— allocated



$P(x_a) = P_{\min}$

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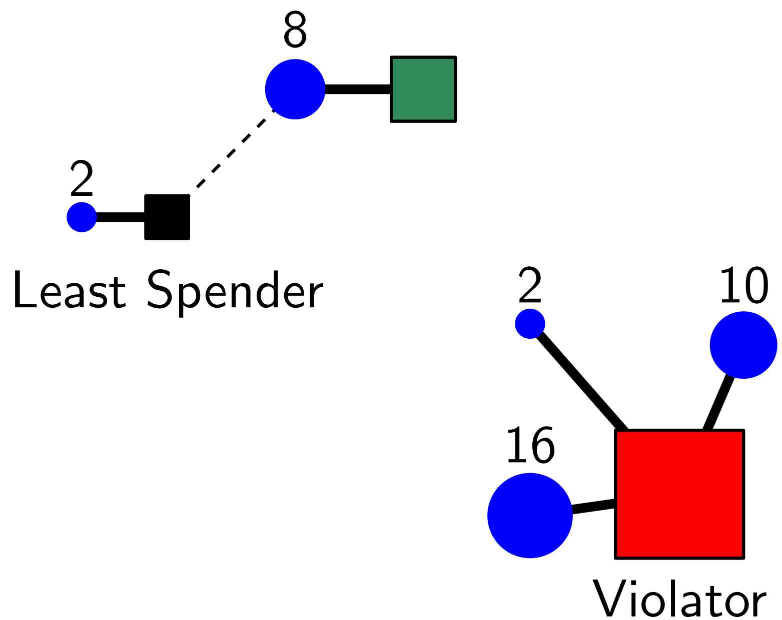
Lower prices to create new swap paths

What if no swap paths from Least Spender to Violator exists?



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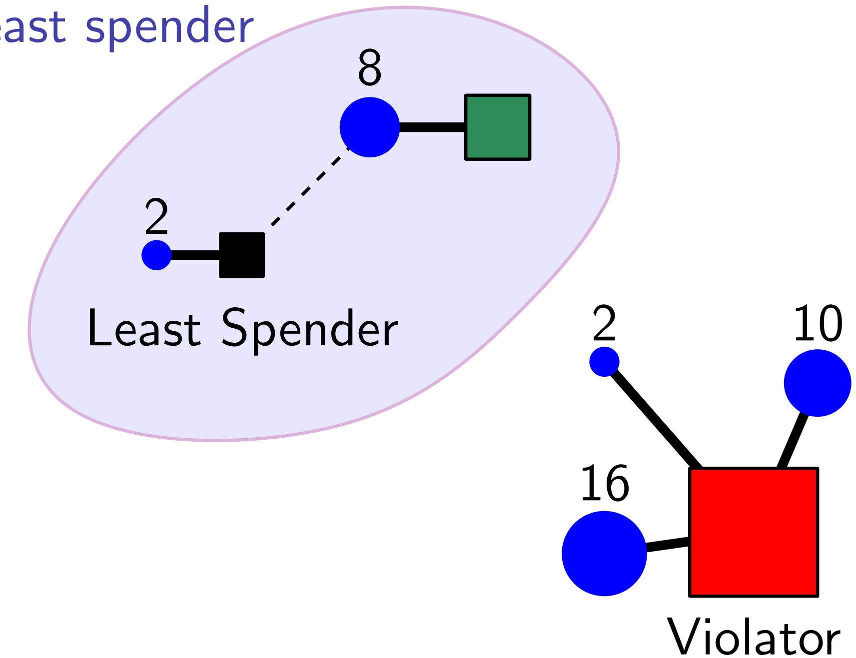
Swap component
can swap towards least spender



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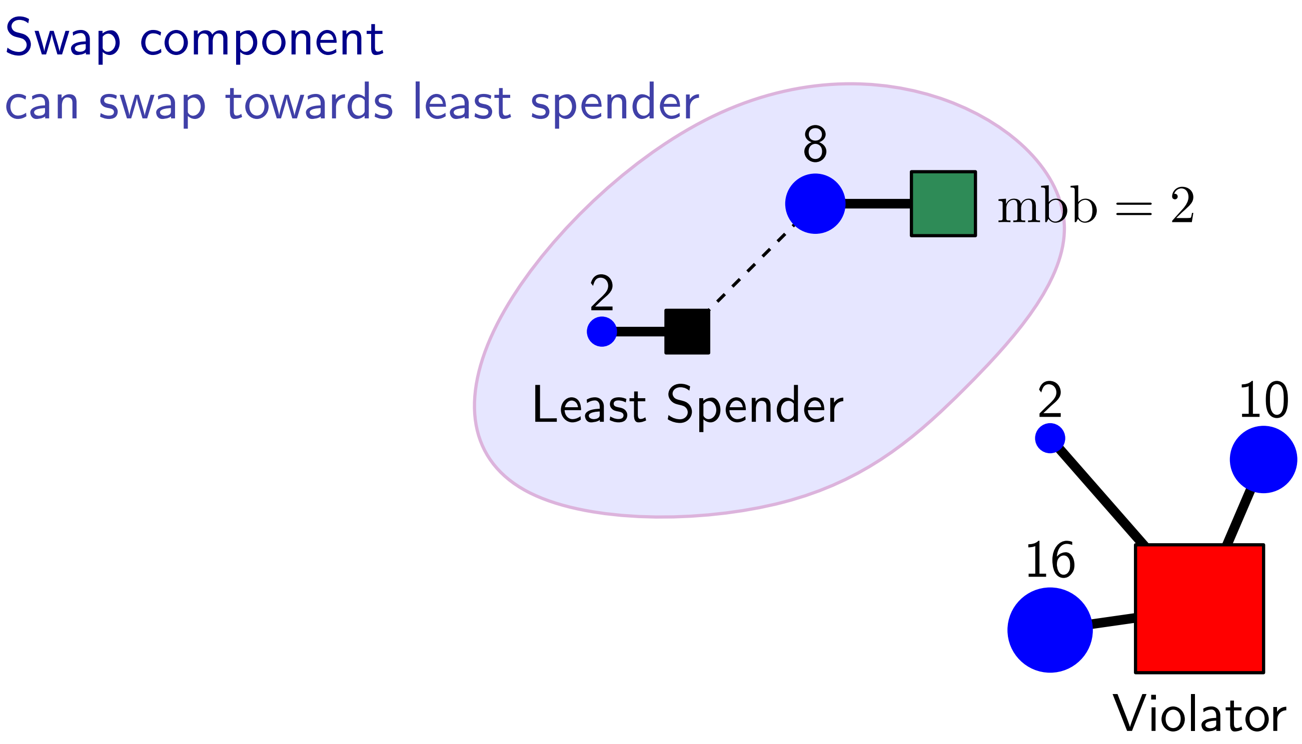

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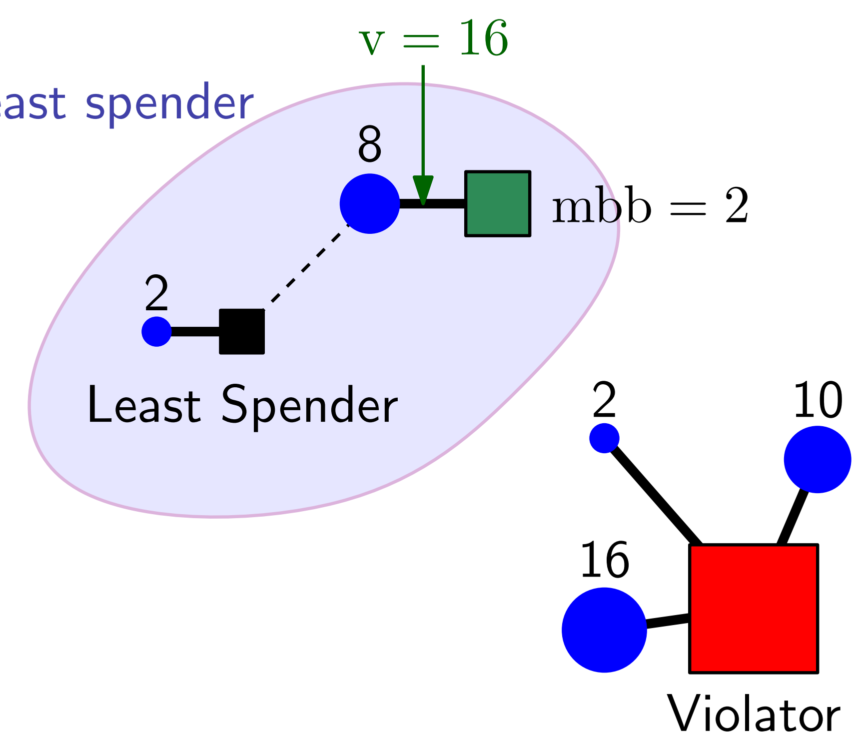

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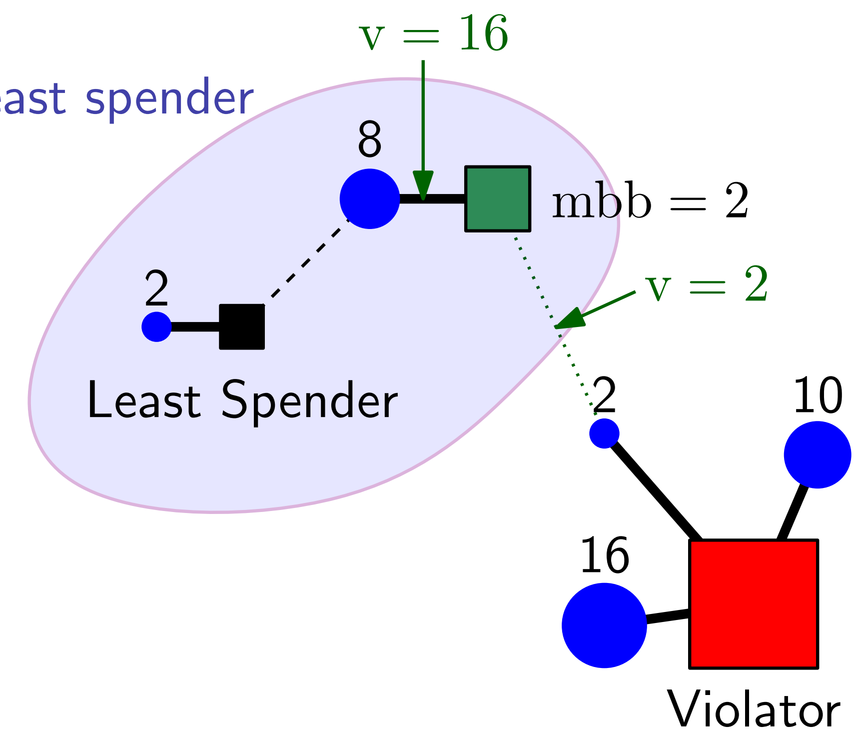

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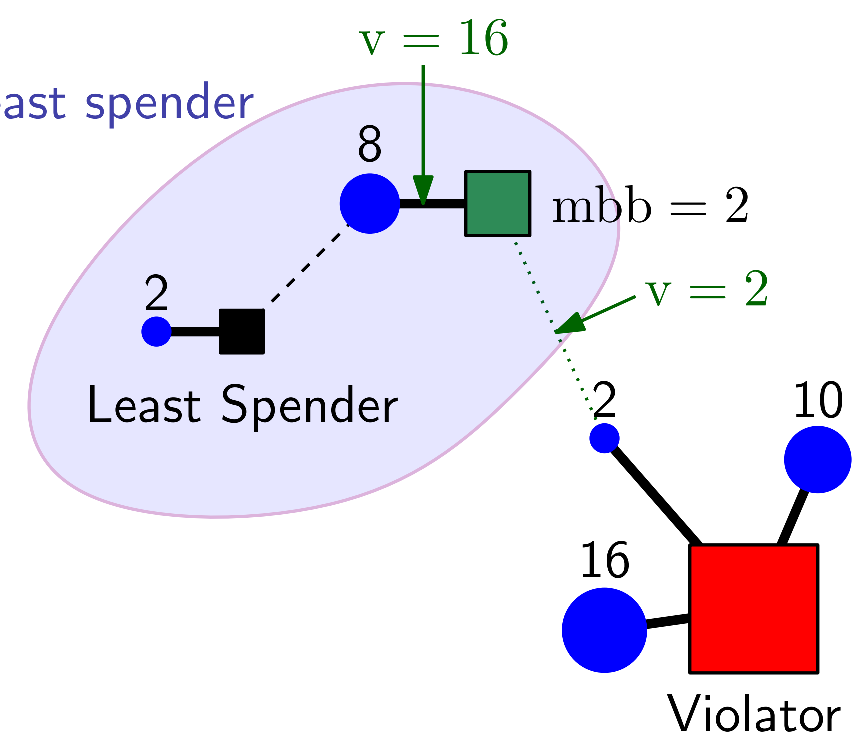

decrease prices to match max-bang-per-buck
$\rightarrow$ grows swap component

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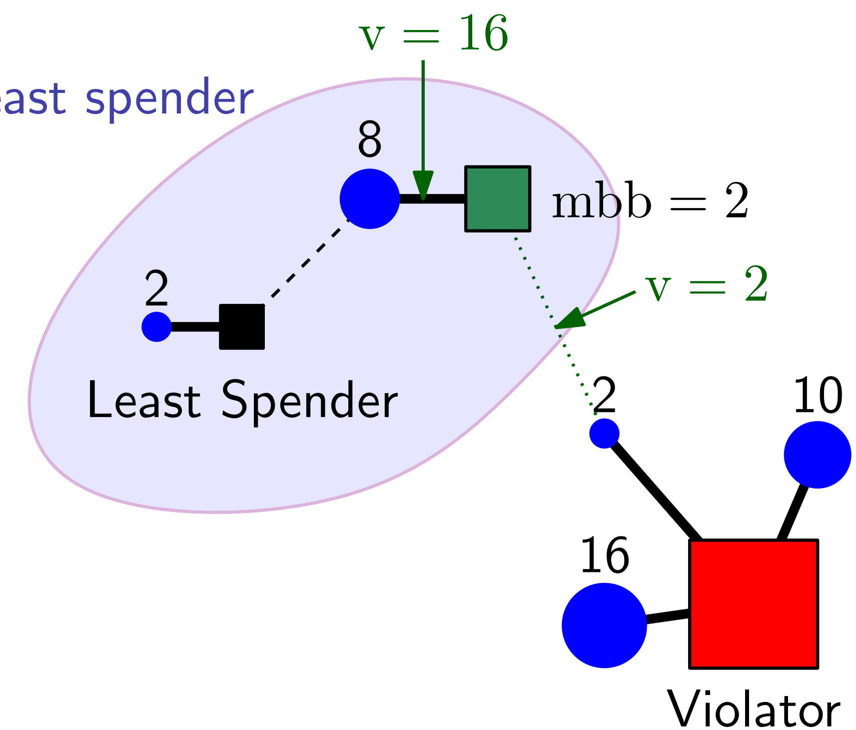


$a \in$ swap component
$g \in$ outside bundle

$$\alpha_{a,g} = \frac{v_{a,g}/p_g}{\mathrm{mbb}_a} < 1$$

multiply by $\max \alpha_{a,g}$

decrease prices to match max-bang-per-buck

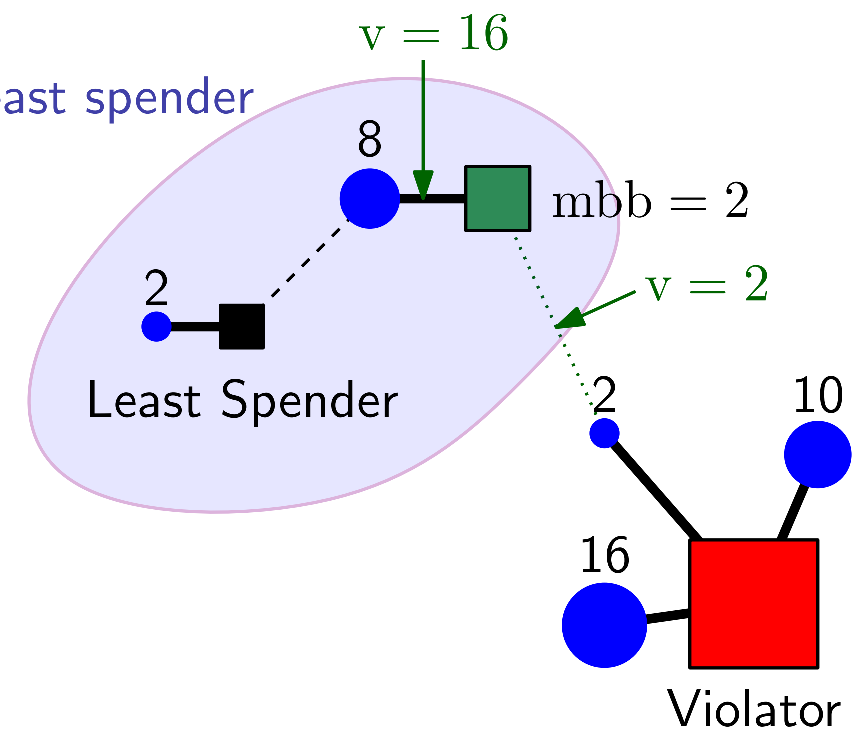
$\rightarrow$ grows swap component

Lower prices to create new swap paths

What if no swap paths from Least Spender to Violator exists?

Swap component

can swap towards least spender



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multiply by $\max \alpha_{a,g}$

below would violate MBB allocation

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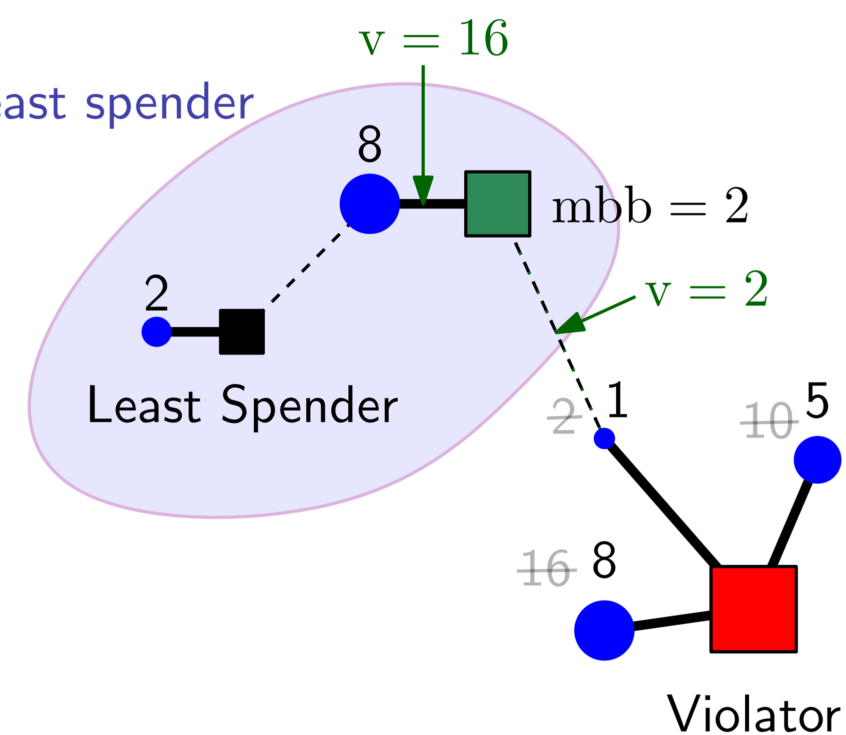

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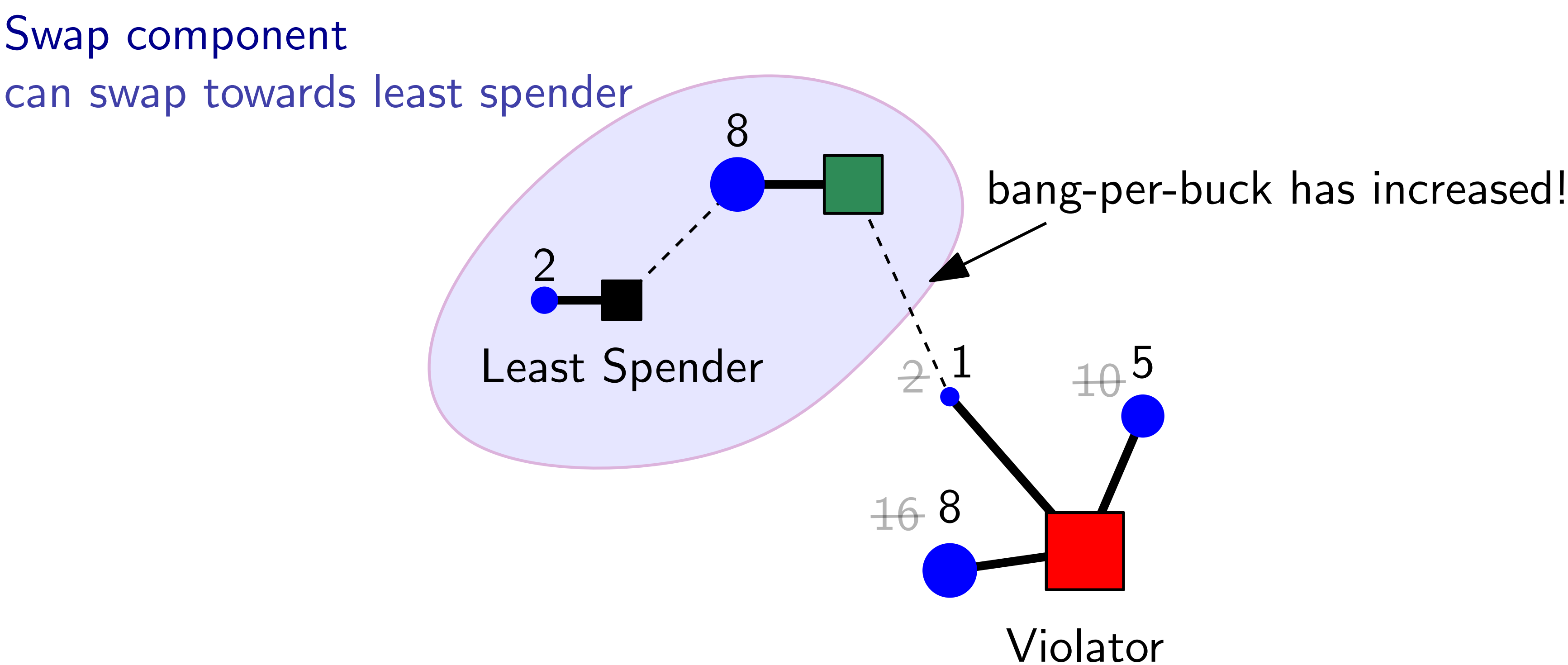

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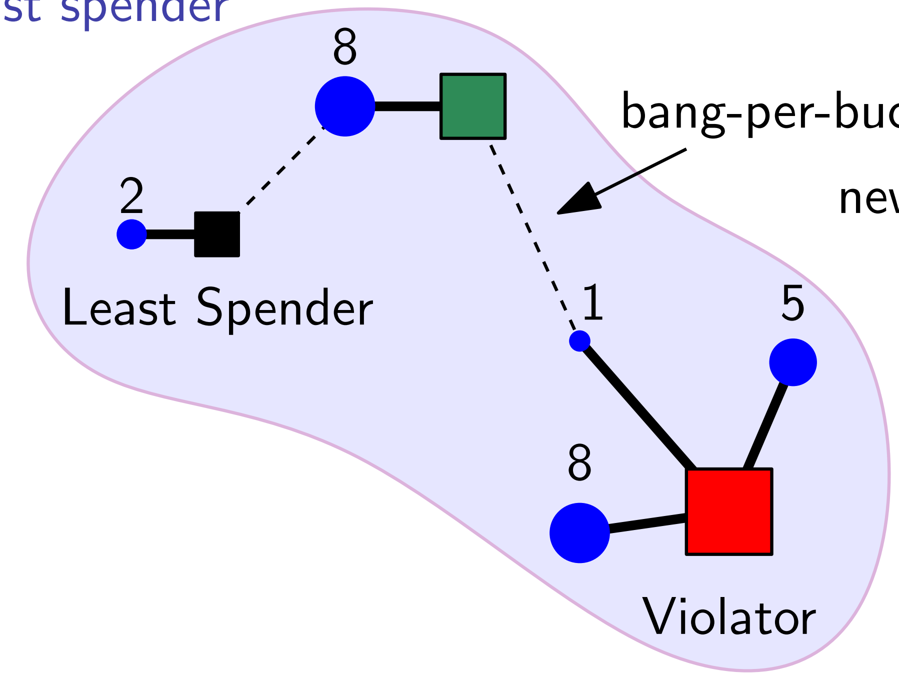

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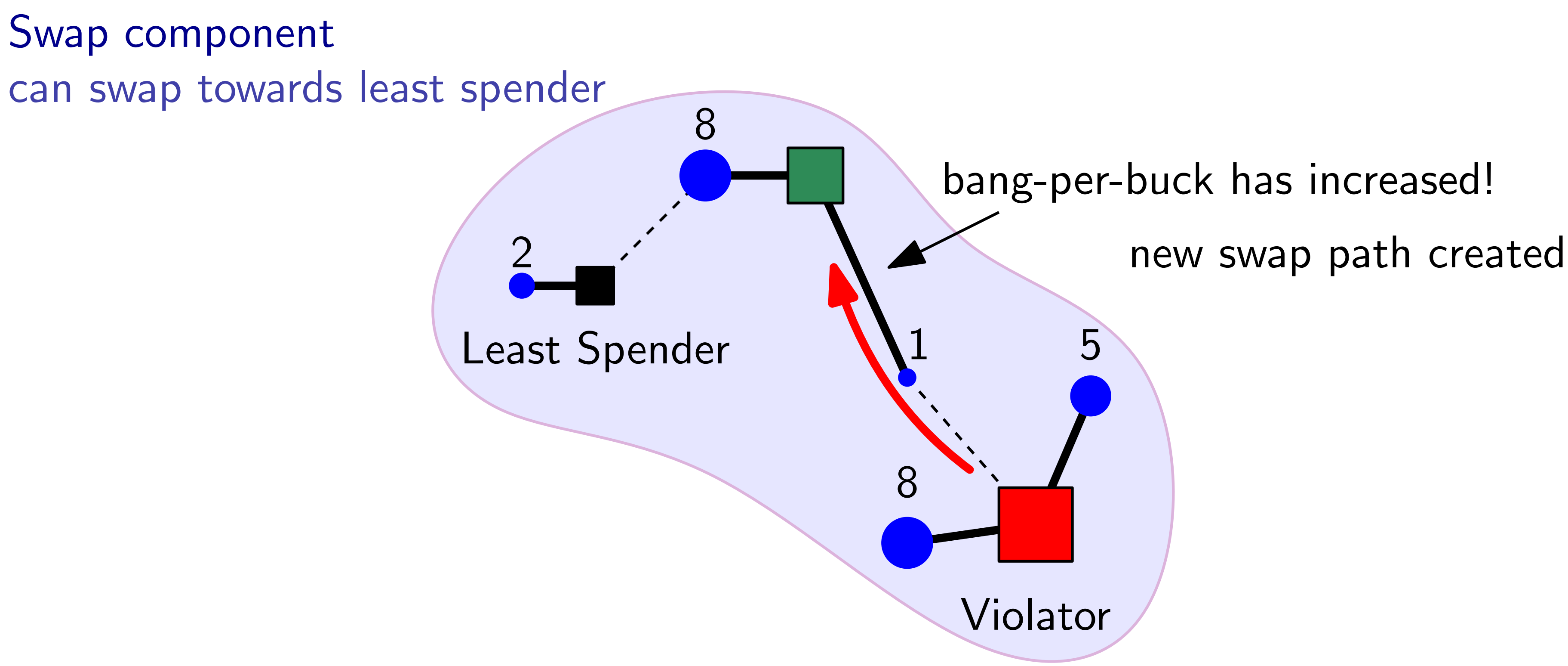

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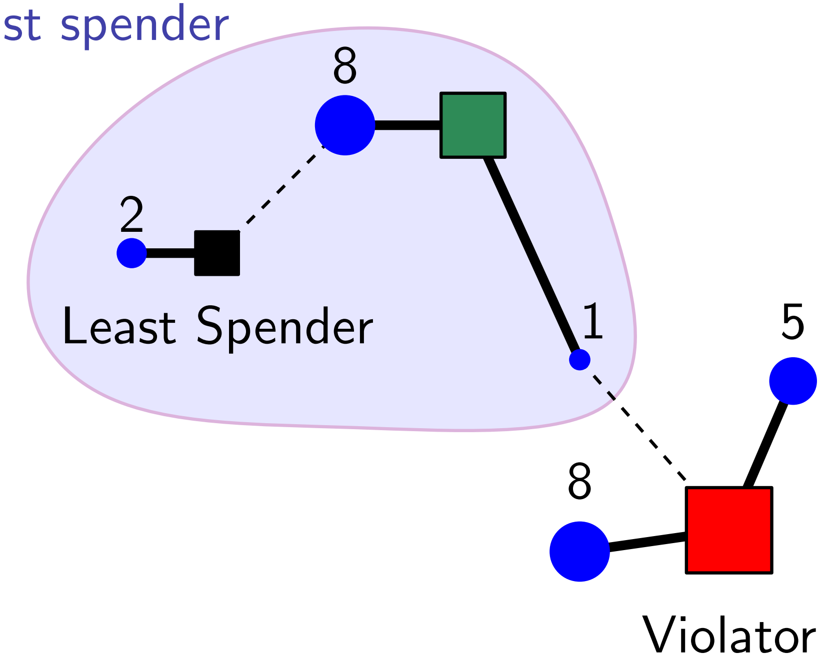

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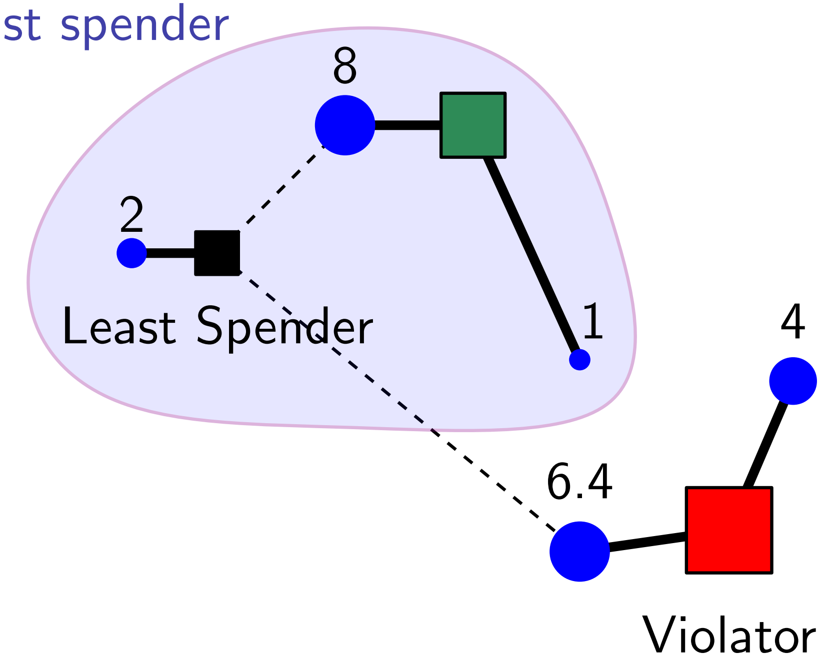

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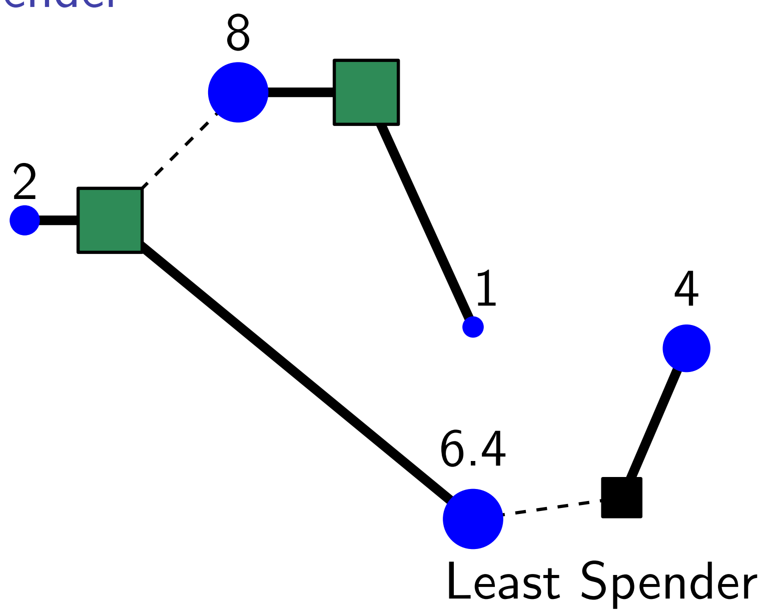

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Least spender's budget does not decrease

budget decreased $\Longrightarrow$ agent was not LS

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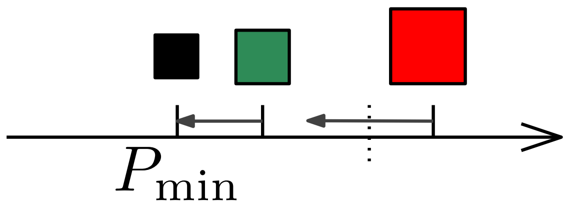
budgets should not drop below LS



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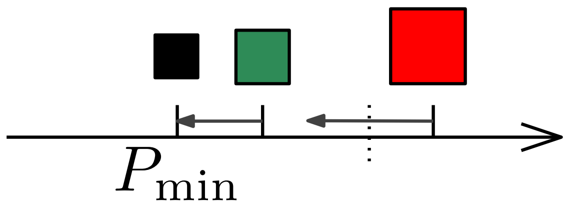

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$$\beta_b = \frac{P(x_b)}{P_{min}} < 1$$

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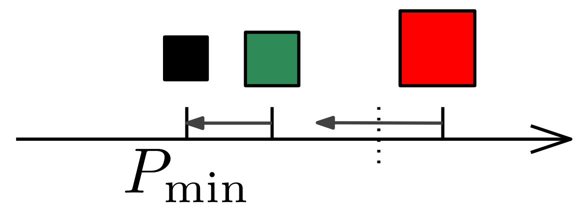

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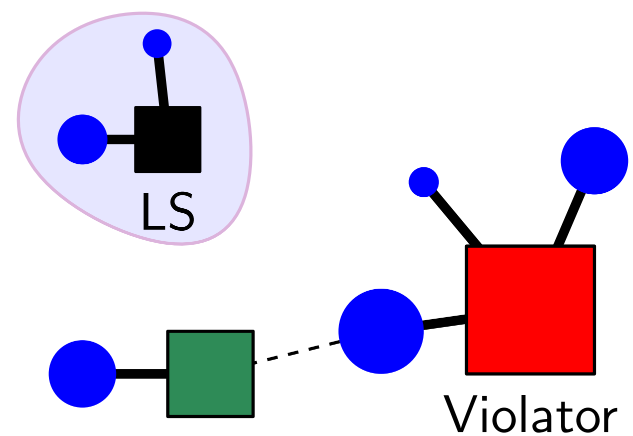

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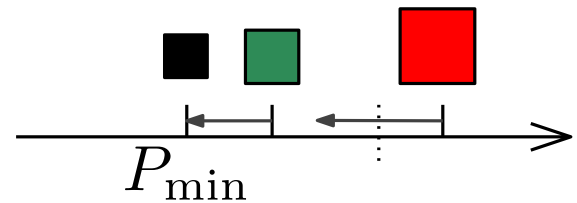

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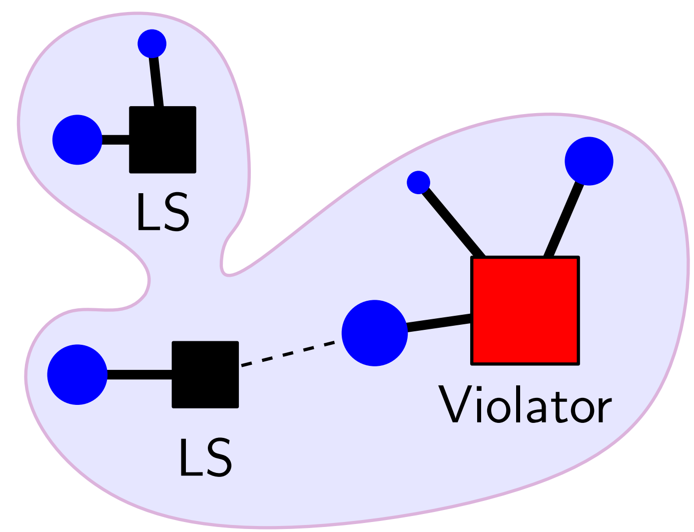

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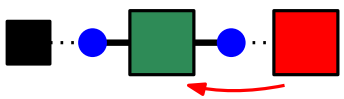
Algorithm outline

`x, p :=` initial complete MBB allocation

`while` pEF1 is violated

  `if` swap path from violator to LS exists

    perform one swap from violator



  `else` grow component

    γ `:=` $\max(\alpha, \beta)$

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    `for` $g \in x_a$ `for` $a \notin S$

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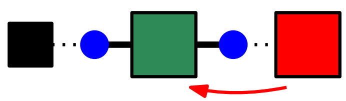

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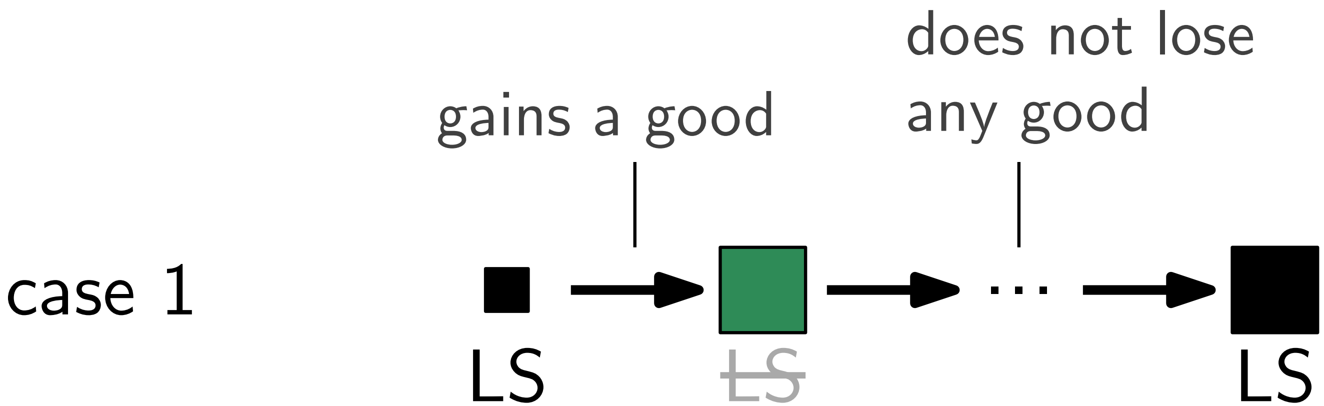
Results

Given any fair division instance with additive valuations, an allocation that is EF1 and PO can be found in $\mathcal{O}(\mathrm{poly}(m, n, v_{\max}))$ time.

For additive valuations, there exists a polynomial-time 1.45-approximation algorithm for the Nash social welfare maximization problem

Value increases every time an agent becomes LS again



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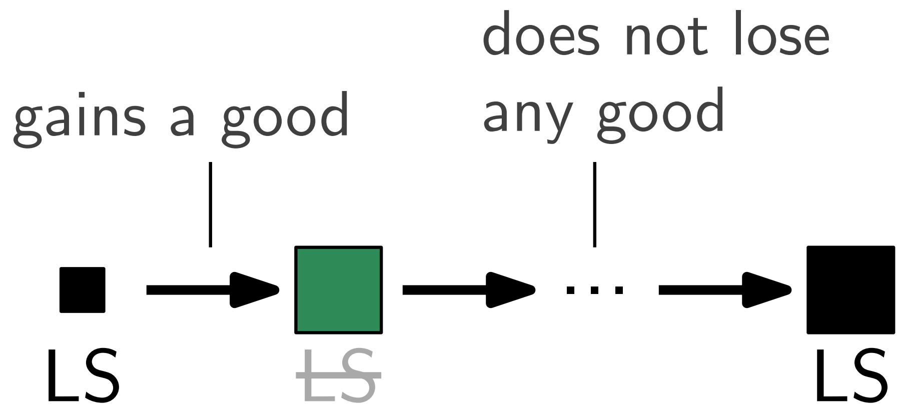

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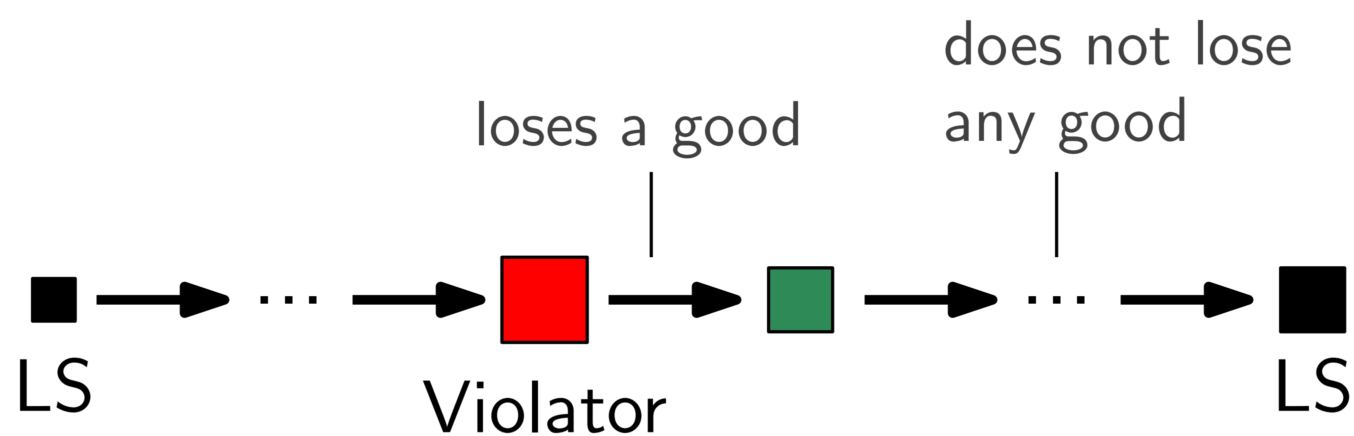

Number of steps with same LS is bounded